# UNIT 12

# *THE GRAZING GOAT*

In this unit, *The Grazing Goat: More Area and Subproblems*, we will revisit many of the ideas that were introduced earlier. In Unit 2 we first introduced the problem solving technique of subproblems, and we return to it here. (We will also continue our work from the last unit on rational expressions.) In Unit 9 we emphasized drawing diagrams to help solve problems. In this unit we will extend the strategy to help solve distance, rate, and time problems. Finally, we will extend our knowledge of solving equations to solve inequalities.

Since many ideas will be coming together in this unit, it is essential that your child clearly shows all diagrams, subproblems, and work for each problem. Organization will be key as the problems become more challenging. Remind your child to keep his Tool Kit organized and up to date as well. Lastly, you might need to be your child's cheerleader as we come closer and closer to the end of the year. We don't want him to give up now! Show confidence in his ability to continue to work hard and be successful.

Let's start with a word problem that can be simplified with the aid of a diagram.

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GG-2. Matilda and Nancy are 60 miles apart, bicycling toward each other on the same road. Matilda rides 12 miles per hour and Nancy rides eight miles per hour. In how many hours will they meet?

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**Solution.**
As suggested, we will draw a diagram to help us solve the problem. In the diagram we should include the information we know. Since Mary is riding faster than Nancy, these two cyclists will not meet in the middle. The will meet closer to Nancy's starting point.

If we consider some specific times, it might make it clearer how to solve this problem. After one hour, Mary has ridden 12 miles while Nancy has ridden 8. They certainly have not met yet since they started 60 miles apart. Nor have they met after two hours when Mary has ridden 24 miles and Nancy has ridden 16. Their total mileage (24 + 16 = 40 miles) does not span the 60 miles. But we can write an equation using x to represent the number of hours that both women have been riding. If Mary rides 12 miles per hour for x hours, then she has ridden a distance of 12x miles. Nancy rides 8 miles per hour for a total of x hours which means she has ridden a distance of 8x miles. Their two distances must add up to the total distance, 60 miles. Thus we have our equation: $12x + 8x = 60$.

$$12x + 8x = 60$$

Solving the equation:

$$20x = 60$$
$$x = 3$$

Therefore the two women will meet after three hours of riding.

**Try it!**

1. Each morning, Tess' father pulls her in her wagon from home to school, averaging 5 miles per hour. At the end of the day, Tess walks the same route home averaging 2 miles per hour. If her total traveling time to and from school is 30 minutes, how far is the school from Tess' home?

**Solution.**

The total distance walking, going and coming, stays the same. The rate and the time change. If Tess walks for 30 minutes, this is the same as walking 0.5 hours (half an hour). If she takes x hours to walk to school, then she takes 0.5 - x hours to return. Now we can write the equation: $5x = 2(0.5 - x)$. Solving for x gives:

$$5x = 1 - 2x$$
$$7x = 1$$
$$x = \frac{1}{7}$$

So it takes Tess' father $\frac{1}{7}$ of an hour to pull Tess in the wagon averaging 5 miles per hour.

That is a distance of $\frac{5}{7}$ miles, or approximately 3771 feet.

This last problem, and other ones like it, are called "distance, rate, time" problems. The key is remembering that distance is equal to the product of the rate and the time; i.e. d = rt. Let's move to rational expressions.

### ADDING AND SUBTRACTING RATIONAL EXPRESSIONS

The Least Common Multiple of (x + 3)(x + 2) and (x + 2) is (x + 3)(x + 2). **Why?**

$$\frac{4}{(x+2)(x+3)} + \frac{2x}{x+2}$$

The denominator of the first fraction already is the Least Common Multiple. To get a common denominator in the <u>second</u> fraction, multiply the fraction by $\frac{x+3}{x+3}$, a form of one (1).

$$= \frac{4}{(x+2)(x+3)} + \frac{2x}{x+2} \cdot \frac{(x+3)}{(x+3)}$$

Multiply the numerator and denominator of the second term:

$$= \frac{4}{(x+2)(x+3)} + \frac{2x(x+3)}{(x+2)(x+3)}$$

Distribute the numerator.

$$= \frac{4}{(x+2)(x+3)} + \frac{2x^2 + 6x}{(x+2)(x+3)}$$

Add, factor, and simplify.

$$= \frac{2x^2 + 6x + 4}{(x+2)(x+3)} = \frac{2(x+1)(x+2)}{(x+2)(x+3)} = \frac{2(x+1)}{(x+3)}$$

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GG-47. Add the following fractions using the same process you used in problem GG-31. Simplify your solution, if possible.

a) $\dfrac{2 - x}{x + 4} + \dfrac{3x + 6}{x + 4}$

b) $\dfrac{x^2 - 4}{x - 5} - \dfrac{x^2 - 4}{x - 5}$

c) $\dfrac{3}{(x + 2)(x + 3)} + \dfrac{x}{(x + 2)(x + 3)}$

d) $\dfrac{3}{x - 1} - \dfrac{2}{x - 2}$

e) $\dfrac{8}{x} - \dfrac{4}{x + 2}$

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**Solution.**

a) $\dfrac{2 - x}{x + 4} + \dfrac{3x + 6}{x + 4} = \dfrac{2 - x + 3x - 6}{x + 4}$

$= \dfrac{2x - 4}{x + 4}$

$= \dfrac{2(x - 2)}{x + 4}$

b) $\dfrac{x^2 - 4}{x - 5} - \dfrac{x^2 - 4}{x - 5} = \dfrac{x^2 - 4 - x^2 + 4}{x - 5}$

$= 0$

c) $\dfrac{3}{(x + 2)(x + 3)} + \dfrac{x}{(x + 2)(x + 3)}$

$= \dfrac{3 + x}{(x + 2)(x + 3)}$

$= \dfrac{1(\cancel{3 + x})}{(x + 2)(\cancel{x + 3})}$

$= \dfrac{1}{x + 2}$

d) $\dfrac{3}{x - 1} - \dfrac{2}{x - 2}$

$= \dfrac{(x - 2)}{(x - 2)} \cdot \dfrac{3}{x - 1} - \dfrac{(x - 1)}{(x - 1)} \cdot \dfrac{2}{x - 2}$

$= \dfrac{3(x - 2)}{(x - 1)(x - 2)} - \dfrac{2(x - 1)}{(x - 1)(x - 2)}$

$= \dfrac{3x - 6 - 2x + 2}{(x - 1)(x - 2)}$

$= \dfrac{x - 4}{(x - 1)(x - 2)}$

e) $\dfrac{8}{x} - \dfrac{4}{x + 2}$

$= \dfrac{(x + 2)}{(x + 2)} \cdot \dfrac{8}{x} - \dfrac{x}{x} \cdot \dfrac{4}{x + 2}$

$= \dfrac{8(x + 2)}{x(x + 2)} - \dfrac{4x}{x(x + 2)}$

$= \dfrac{8x + 16 - 4x}{x(x + 2)}$

$= \dfrac{8x + 16 - 4x}{x(x + 2)}$

$= \dfrac{4x + 16}{x(x + 2)}$

$= \dfrac{4(x + 4)}{x(x + 2)}$

**Try it!**

2. Add or subtract and simplify.

a) $\dfrac{6}{x(x - 3)} + \dfrac{2}{x + 3}$  b) $\dfrac{3x + 1}{x^2 - 16} - \dfrac{3x + 5}{x^2 + 8x + 16}$

**Solution.**

2. a) $\dfrac{6(x + 3)}{x(x - 3)(x + 3)} + \dfrac{2(x)(x - 3)}{x(x + 3)(x - 3)} = \dfrac{2x^2 + 18}{x(x - 3)(x + 3)} = \dfrac{2(x^2 + 9)}{x(x - 3)(x + 3)}$

b) $\dfrac{3x + 1}{(x + 4)(x - 4)} - \dfrac{3x + 5}{(x + 4)(x + 4)} = \dfrac{(3x + 1)(x + 4)}{(x - 4)(x + 4)^2} - \dfrac{(3x + 5)(x - 4)}{(x - 4)(x + 4)^2}$

$= \dfrac{20x + 24}{(x - 4)(x + 4)^2} = \dfrac{4(5x + 6)}{(x - 4)(x + 4)^2}$

In this unit we will apply our experience with solving equations to solving inequalities. The methods are similar; we simply make small adjustments to accommodate the fact that we have many, many more (infinitely many more!) solutions possible for inequalities.

**INEQUALITY NOTATION**

| **Symbol** | **Translation** |
|---|---|
| $<$ | less than |
| $>$ | greater than |
| $\le$ | less than or equal to |
| $\ge$ | greater than or equal to |

**SOLVING INEQUALITIES**

To solve an inequality, we first treat the problem as an equality. The solution to the equality is called the **DIVIDING POINT**. For example, $x = 12$ is the Dividing Point of the inequality $3 + 2(x - 5) \le 17$, as shown below:

Problem: $3 + 2(x - 5) \le 17$

First change the problem to an equality and solve for x:

$$\begin{gathered} 3 + 2(x - 5) = 17 \\ 3 + 2x - 10 = 17 \\ 2x - 7 = 17 \\ 2x = 24 \\ x = 12 \end{gathered}$$

Since our original inequality <u>included</u> $x = 12$, we place the Dividing Point on the number line as a solid point. We then test one value on either side of the Dividing Point in the <u>original</u> inequality to determine which set of numbers makes the inequality true.

| **Test: x = 8** | **Test: x = 15** |
|---|---|
| $3 + 2(\mathbf{8} - 5) \le 17$ | $3 + 2(\mathbf{15} - 5) \le 17$ |
| $3 + 2 \cdot 3 \le 17$ | $3 + 2 \cdot 10 \le 17$ |
| $3 + 6 \le 17$ | $3 + 20 \le 17$ |
| **TRUE!** | **FALSE!** |

Therefore, the solution is $x \le 12$.

When the inequality is "<" or ">," then the Dividing Point is <u>not</u> included in the answer. On a number line, this would be indicated with an open circle at 12.

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GG-56. Use the method from problem GG-55 to solve the following inequalities for x.

a) $4x - 1 \geq 7$

b) $2(x - 5) \leq 8$

c) $3 - 2x < x + 6$

d) $\frac{1}{2}x > 5$

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**Solution.**

a) $4x - 1 \geq 7$. Dividing point: $4x - 1 = 7$
$4x = 8$
$x = 2$

Let's test x = 0 in the original inequality:

$4(0) - 1 \overset{?}{\geq} 7$ FALSE!

That means the <u>other</u> side of the Dividing Point is true. Our solution is $x \geq 2$.

b) $2(x - 5) \leq 8$. Dividing point: $2(x - 5) = 8$
$x - 5 = 4$
$x = 9$

Let's test x = 0 again.

$2(0 - 5) \overset{?}{\leq} 8$

$-10 \leq 8$ TRUE! Shade the side of the Dividing Point that zero is on. The solution is $x \leq 9$.

c) $3 - 2x < x + 6$. Dividing point:
$3 - 2x = x + 6$
$3 = 3x + 6$
$-3 = 3x$
$x = -1$

Let's test x = 0 in the original inequality:

$3 - 2(0) \overset{?}{<} 0 + 6$

$3 < 6$ TRUE! Shade the side of the Dividing Point that zero is on. Our dividing point has an open circle since the inequality is strictly less than. Our solution is $x > -1$.

d) $\frac{1}{2}x > 5$. Dividing point: $\frac{1}{2}x = 5$
$x = 10$

Testing x = 0 we find

$\frac{1}{2}(0) \overset{?}{>} 5$

$0 > 5$ FALSE! Shade the opposite side of the Dividing Point from 0. The solution is $x > 10$.

**Try it!**

3. Solve the following inequalities for x.

a) $3(x + 4) > 12$ b) $2x - 7 \leq 5 - 4x$

**Solution.**

3. a) $x > 0$ b) $x \leq 2$

Graphing inequalities extends to other areas as well. We will now see how to graph quadratic inequalities as well as inequalities containing absolute values by doing a few of them.

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**GG-66.** In problem GG-55 we focused on linear inequalities. How can we use those ideas with non-linear inequalities? In this problem, we will focus on the quadratic inequality at right.

$$x^2 - 3x - 18 \le 0$$

a) To find our dividing point(s), solve the equality $x^2 - 3x - 18 = 0$.

b) In the case of this parabola, we seem to have **two** dividing points! Plot these dividing points on a number line.

c) How many regions (sections or "pieces") do two different points create on the number line? Identify each region.

d) For what x-values is $x^2 - 3x - 18 \le 0$? Test a number in each region. Be sure to consider whether the dividing points themselves are part of the solution. e) Suppose we change the direction of the inequality symbol. Solve: $x^2 - 3x - 18 > 0$.

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**Solution.**

Just as before, we first find our "dividing point." The equation to solve here happens to be a quadratic and so we can solve this by factoring or using the Quadratic Formula. This one happens to factor as shown at right.

$$x^2 - 3x - 18 = 0$$
$$(x - 6)(x + 3) = 0$$
$$x = 6 \quad \text{or} \quad x = -3$$

Since the equation has two solutions, the graph has two dividing points. Each is a solid circle since the original inequality is "≤". By looking at the graph we can see that these two points divided the line into three different sections or pieces. Now, just as before, we can test points within the sections to determine if the points will make the inequality true or false. I will test the points x = -4, x = 0 and x = 7, one point from each section, in the original inequality.

| | | |
|---|---|---|
| $x^2 - 3x - 18 \le 0$, $x = -4$ | $x^2 - 3x - 18 \le 0$, $x = 0$ | $x^2 - 3x - 18 \le 0$, $x = 7$ |
| $(-4)^2 - 3(-4) - 18 \overset{?}{\le} 0$ | $(0)^2 - 3(0) - 18 \overset{?}{\le} 0$ | $(7)^2 - 3(7) - 18 \overset{?}{\le} 0$ |
| $16 + 12 - 18 \overset{?}{\le} 0$ | $0 - 0 - 18 \overset{?}{\le} 0$ | $49 - 21 - 18 \overset{?}{\le} 0$ |
| $10 \le 0$ FALSE! | $-18 \le 0$ TRUE! | $10 \le 0$ FALSE! |

On the graph, the result of our testing looks like the figure at right.

The solution is $-3 \le x \le 6$.

To answer part (e), we do not need to completely redo the problem. The dividing points will be the same but for the inequality $x^2 - 3x - 18 > 0$ they will be open circles. Using the same test points, we can examine the work above to find the regions that make the quadratic expression greater than 0. That happens in the first and last region, leaving the middle section as the "false" region. In this case, the solution is $x < -3$ or $x > 6$ (see graph above).

**Try it!**

4. Solve each inequality for x by finding the dividing points and testing a point in each region.

a) $x^2 - 3x - 4 < 0$ b) $-x^2 - 9x - 14 < 0$

**Solution.**

4. a) $-1 < x < 4$ b) $x < -7$ or $x > 2$

When solving an inequality which contains an absolute value of a variable or expression, we can proceed just as we have done so far: find the dividing points then test numbers in each region to determine whether the region makes the original inequality true or false. We can also, however, use the definition of absolute value to create a geometric solution. Let's look at both methods; you decide which method you prefer.

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GG-83. Solve each inequality for x:

a) $|x| - 4 > 1$

b) $|x - 4| > 1$

c) Compare parts (a) and (b) and their solutions. What do you notice? Is there any connection?

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**Solution.**

Let's solve part (a) algebraically. As we have said, we first treat this as an equality and solve to find the dividing points. Remember that when solving equations containing absolute values algebraically, the one equation becomes two that are solved. In this problem we have two dividing points ($x = -5$, $x = 5$) and must check one point from each of the three regions in the original inequality. I will use $x = -6$, $x = 0$, and $x = 6$ as my test points.

$$|x| - 4 = 1$$
$$|x| = 5$$
$$x = 5 \text{ or } x = -5$$

| | | |
|---|---|---|
| $\|x\| - 4 > 1,\ x = -6$ | $\|x\| - 4 > 1,\ x = 0$ | $\|x\| - 4 > 1,\ x = 6$ |
| $\|-6\| - 4 \overset{?}{>} 1$ | $\|0\| - 4 \overset{?}{>} 1$ | $\|6\| - 4 \overset{?}{>} 1$ |
| $6 - 4 > 1$ TRUE! | $0 - 4 > 1$ FALSE! | $6 - 4 > 1$ TRUE! |

Therefore the solution is $x < -5$ or $x > 5$.

In part (b) we will use a geometric approach to solve this inequality. Recall that the definition of absolute value is the distance a point is from zero -- or perhaps the distance from another point when the expression within the absolute value bars is changed. This is the case in part (b). Here the problem can be translated to a geometric interpretation as: we want the points with distances from 4 to be greater than 1. That would be the points greater than 5 or the points less than 3, i.e. $x > 5$ or $x < 3$.

Part (c) asks for us to notice any connections. Both inequalities share a common region in their solutions: $x > 5$. Since the inequalities are similarly written, we should expect some similarity in their solutions.

**Try it!**

5. Solve each inequality for x.

a) $|x + 6| \leq 11$ b) $|x| + 4 > 2$

**Solution.**

5. a) $-5 \leq x \leq 17$ b) All real numbers are solutions, since $|x|$ is positive and 4 is already greater than 2.

Inequalities can also be solved in the x-y plane. It is very similar to graphing lines. Let's try one.

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GG-102. On graph paper, graph and shade the solution for each of the systems of inequalities below. Describe each resulting region.

a) $y \le \frac{2}{5}x$

$y > 5 - x$

b) $y \le \frac{1}{4}x + 3$

$y \ge \frac{1}{4}x - 2$

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**Solution.**

We will graph each inequality for part (a) on the same set of axes. First, we treat the inequality as an equation: $y = \frac{2}{5}x$. We can graph it using the slope, $\frac{2}{5}$, and the y-intercept, (0, 0). This line divides the grid into two regions. As before, we choose a point in either region and check whether or not it makes the original inequality true. In this example I'll use the point (0, 1) as the test point.

$$1 \overset{?}{\le} \frac{2}{5}(0) \quad \text{FALSE!}$$

Since points above the line make the inequality false, points below the line must make it true. Therefore we shade all the points below the line to represent the solution.

Next we do the same thing for the second inequality. First treat it as an equation and graph it on the same set of axes. The slope is -1 and the y-intercept is (0, 5). This line is dashed (not solid) because the inequality is strictly greater than, not greater than or equal to. The line has divided the grid into two regions and we will chose a point on one side as a test point. I will use the same point as last time. (This is not necessary and may not always be feasible. Here it is a convenient point.)

$$1 \overset{?}{>} 5 - 0 \quad \text{FALSE!}$$

Since our test point made the inequality false, the opposite side would make the inequality true. For the second inequality we shade above the line and to the right.

Putting these two inequalities together with their overlapping shading gives us the solution to the system of inequalities. In this case the solution is the darkest region, with a solid line above the region and a dashed line bordering below (left).

Part (b) is done similarly by first treating the inequalities as equations and graphing the lines by using their slopes and y-intercepts. Both equations have a slope of $\frac{1}{4}$ ; the y-intercepts are different; both lines are solid.

Next we use a test point in the original inequalities. Here, (0, 0) is a simple point and works well for both inequalities.

$$0 \overset{?}{\leq} \frac{1}{4}(0) + 3 \quad \text{TRUE!}$$

$$0 \overset{?}{\geq} \frac{1}{4}(0) - 2 \quad \text{TRUE!}$$

Since (0, 0) makes both inequalities true, we shade below the top line and above the bottom line. The result is that the region between the two lines is shaded. This gives us the solution shown below.

**Try it!**

6. Graph and shade the solution for each system of inequalities below.

a) $y < \frac{3}{5}x - 2$

$y < -3x + 4$

b) $y \leq 3x + 2$

$y > -\frac{1}{3}x - 4$

**Solution.**

a)

b)

This unit also offers some rich problems in which the students can practice all of their problem solving techniques. Throughout this year, some techniques have been employed often (Guess and Check, for example).

**PROBLEM SOLVING STRATEGIES**

- Making a Guess and then Checking It (Guess And Check)
- Using Manipulatives such as Algebra Tiles
- Making Systematic Lists
- Graphing a Situation
- Drawing a Diagram
- Breaking a large problem into smaller Subproblems
- Writing and Solving an Equation

In problems GG-114 through GG-120 you will encounter a variety of problems that will require a problem solving strategy to solve. Remember to add new problem solving strategies to your tool kit as they are developed.

To illustrate two of the techniques (Drawing a Diagram and Breaking a large problem into smaller Subproblems) let's solve the theme problem for this unit. The Grazing Goat is challenging, but the students are well-prepared for it. Patience and organization are also important here.

?*Behind the Problem*

Read through the problem (GG-127 below) completely with your child. Ask her what the problem is about. Usually what students find difficult with this problem is not the calculations (although they need to be careful to avoid careless mistakes), but determining the shape of the goat's grazing area. Your child must be convinced that a rope attached at one fixed point will sweep out a circle. It might help to take a piece of string and a thumbtack and tack the string down. Move the other end, keeping the string taunt, and a circle is formed. If the rope or string is attached to the corner of a building what happens? The rope wraps around the building and only part of a circle is formed. Putting a box or block at the tack will help your child visualize this.

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GG-127. THE GRAZING GOAT

Zoe the goat is tied by a rope to one corner of a 15 meter by 25 meter barn in the middle of a large, grassy field. Over what area of the field can Zoe graze if the rope is:

a) 10 meters long? c) 30 meters long? b) 20 meters long? d) 40 meters long? e) Zoe is happiest when she has at least 400 $m^2$ to graze. What possible lengths of rope could be used?

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**Solution.**
First we draw a sketch. At right it shows that the rope sweeps out almost a complete circle -- how much of a circle is it? It is three-fourths of the circle. But the drawing here is with a rope less than 15 meters long. How do we know the rope in this picture is less than 15 meters long? If it were 15 meters, the rope would reach to the lower left-hand corner of the barn. Any longer and the rope would wrap around that same corner, touching the other 25 meter side.

So in part (a), we want the rope to be 10 meters long. The picture above can show this. The area for Zoe is three-fourths of a circle with radius 10 meters. The area of a complete circle is $A = \pi r^2$, so the area of this region is $\frac{3}{4}\pi(10)^2 = 75\pi \approx 235.62$ square meters.

In part (b) we want a rope 20 meters long. What will this do to the picture? Now the region will wrap around the barn. Looking closely you can see that we still have three-fourths of a circle (this time with a radius of 20 meters) but we also have a small quarter of a circle. What is the radius of the small quarter circle? Since the rope is 20 meters long and the side of the barn is only 15 meters long, that leaves 5 meters of rope to wrap around the barn. Thus we need to find two areas:

The area of a complete circle is found with the formula $A = \pi r^2$. The first piece is $\frac{3}{4}$ of a circle while the second piece is $\frac{1}{4}$ of a different circle. We can find the total area as follows:

$$A = \frac{3}{4}\pi(20)^2 + \frac{1}{4}\pi(5)^2$$

$$A = \frac{1200\pi}{4} + \frac{25\pi}{4}$$

$$= \frac{1225\pi}{4}$$

$$= 306.25\pi$$

$$\approx 962.11 \text{ square meters.}$$

The goat has approximately 962.11 square meters of grazing room in this case.

In part (c) we have a bigger problem. Now the rope will wrap around both sides as shown:

To find the area of this shaded region we have to find the area of three separate parts. One is three-fourths of a circle with a 30 meter radius, one is one-fourth of a circle with a 15 meter radius and the last is one fourth of a circle with a five meter radius.

The calculation looks like this:

$$\begin{aligned} A &= \frac{3}{4}\pi(30)^2 + \frac{1}{4}\pi(15)^2 + \frac{1}{4}\pi(5)^2 \\ &= \frac{2700\pi}{4} + \frac{225\pi}{4} + \frac{25\pi}{4} \\ &= \frac{2950\pi}{4} \\ &= 737.5\pi \\ &\approx 2316.92 \text{ square meters.} \end{aligned}$$

Here the goat has approximately 2316.92 square meters of field to graze in.

In part (d) something interesting happens. Yes, the rope will wrap around both sides as shown at right. But because of the rope length, the goat can just reach the lower right-hand corner of the barn from either direction. But, you can see that it is not a complete circle of radius 40. In fact, we still have three fourths of one circle (with radius 40 meters), one-fourth of a circle with radius 25 meters and another one-fourth of a circle with radius 15 meters.

The calculations look like this:

$$\begin{aligned} A &= \frac{3}{4}\pi(40)^2 + \frac{1}{4}\pi(25)^2 + \frac{1}{4}\pi(15)^2 \\ &= \frac{4800\pi}{4} + \frac{625\pi}{4} + \frac{225\pi}{4} \\ &= \frac{5650\pi}{4} \\ &= 1412.5\pi \\ &\approx 4437.50 \text{ square meters.} \end{aligned}$$

Here the goat has approximately 4437.50 square meters of grazing area.

Part (e) tells us that Zoe is happiest when she has at least 400 square meters of grazing area. By looking back at our answers we can see that the rope must be more than 10 meters to achieve this, but can be less than 20 meters as well. You can use several approaches here. One possibility is to use Guess and Check to find the value. (If you do this, don't start at 11 and work up, or at 19 and work down. Try 15 first. After all, if the rope is longer than 15 meters, we have a different set up than if the rope is less than or equal to 15 meters.)

To solve this, I will first consider the easier case of the rope being less than or equal to 15 meters. Then there would be only one area to find: three-fourths of a circle. Let's label the radius of this circle x. The area we are looking for would be $A = \frac{3}{4}\pi x^2$. We want to have this greater to or equal to 400.

$$\begin{aligned} &400 \geq \frac{3}{4}\pi x^2 \\ &\frac{4}{3}(400) \geq \pi x^2 \\ &\frac{4}{3\pi}(400) \geq x^2 \\ &x \geq \sqrt{\frac{4}{3\pi}(400)} \geq 13.03 \text{ meters.} \end{aligned}$$

Good! The rope does not need to wrap around the barn. Note: if we had found x to be greater than 15, then we would have written a more complicated equation where we would sum the area of three-fourths of a circle and the area of a quarter circle of radius (x - 15).

**Try it!**

7. Solve the Grazing Goat problem with a rope 25 meters long.

**Solution.**

7. $493.75\pi \approx 1551.16$ square meters.

This unit does have more challenging problems which require various problem solving strategies. Be sure you work as many problems as you can. Do remember: you can learn just as much—if not more—from a problem you can't do as from a problem you can do.

# More to Try

Draw a diagram, write an equation, and solve.

1. Agatha and Barry start at the same point and drive in opposite directions. Agatha drives 50 miles per hour while Barry drives 55 miles per hour. How far apart will they be after four hours?

2. Conrad and Davis start at the same point and drive in opposite directions. Conrad drives 35 miles per hour while Davis drives 45 miles per hour. How far apart will they be after two hours?

3. Eden and Farley start at the same point and drive in opposite directions. Eden drives 60 miles per hour while Farley drives 65 miles per hour. How far apart will they be after five hours?

4. Gloria and Hank start at the same point and drive in opposite directions. Gloria drives 45 miles per hour while Hank drives 55 miles per hour. How far apart will they be after six hours?

5. Iris and Jerry start at the same point and head in opposite directions. Iris walks due north at 2.5 miles per hour and Jerry bikes due south at 8 miles per hour. How far apart will they be after four hours?

6. Walt and Yani start at the same point and head in different directions. Walt sails west at 12 miles per hour while Yani sails south at 14 miles per hour, how far apart will they be after ten hours?

7. Cassie and Devon start at the same point and head in different directions. Cassie's motor boat travels east at 34 miles per hour while Devon's motor boat travels south at 27 miles per hour. How far apart will they be after four hours?

Simplify.

8. $\frac{7x - 1}{x^2 - 2x - 3} - \frac{6x}{x^2 - x - 2}$

9. $\frac{3}{x - 1} + \frac{4}{1 - x} + \frac{1}{x}$

10. $\frac{3y}{9y^2 - 4x^2} - \frac{1}{3y + 2x}$

11. $\frac{2}{x + 4} - \frac{x - 4}{x^2 - 16}$

12. $\frac{5x + 9}{x^2 - 2x - 3} + \frac{6}{x^2 - 7x + 12}$

13. $\frac{x + 4}{x^2 - 3x - 28} + \frac{x - 5}{x^2 + 2x - 35}$

Solve the following inequalities. Include a number line graph of the solution.

14. $3x + 2 < 11$

15. $4(x - 6) \geq 20$

16. $\frac{1}{4} x < 2$

17. $12 - 3x > 2x + 1$

18. $\frac{x - 5}{7} \leq -3$

19. $3(5 - x) \geq 7x - 1$

20. $3x^2 + 7x - 6 \geq 0$

21. $x^2 + 4x - 8 < 4$

22. $x(7x - 26) \leq 8$

23. $|x + 4| \geq 7$

24. $|x| - 5 \leq 8$

25. $-2|x - 3| + 6 < -4$

Graph and shade the solution for the system of inequalities below.

26. $y \leq -x + 2$
$y \leq 3x - 6$

27. $y > \frac{2}{3}x + 4$
$y < \frac{7}{12}x + 5$

28. $x < 3$
$y \geq -2$

29. $y \leq 4x + 16$
$y > -\frac{4}{3}x - 4$

30. Tess can spread toys over the floor of a room, covering the floor at a rate of 9 square feet per minute. Davis can cover the floor with toys at a rate of 6 square feet per minute. If they both decide to play in a room with 180 square feet of floor space, how long will it take to cover the floor, playing together?

31. A 20 inch pizza (that means the diameter is 20 inches) is cut into 12 slices, perfectly, so that each slice is the same size. Five slices are eaten. What is the area of the remaining pizza?

32. An eight-inch pizza is cut into fourths and only one piece is eaten. What is the area of the remaining pizza?

33. In the figure at right, the circles fit perfectly inside the square with side length four inches. There is no overlap of the circles (i.e. the circles are tangent to each other and to the square.) What is the area of the shaded region?

Write an equation to solve these problems.

34. Three numbers are in the ratio of 2 : 3 : 3. What are the numbers if their sum is 32 ?

35. Three numbers are in the ratio of 4 : 10 : 12. What are the numbers if their sum is 182 ?

36. Three numbers are in the ratio of 3 : 6 : 9. What are the numbers if their sum is 270 ?

37. Three numbers are in the ratio of 7 : 10 : 11. What are the numbers if their sum is 84 ?