Exponential Functions

GUIDED SKILL SUPPORT

Overview

Students develop their understanding of exponential functions from geometric sequences. They examine multiple representations of exponential functions, including graphs, tables, equations, and contexts. They learn how to move from one representation to another. In exponential functions of the form $f(x) = ab^x$, students learn that the value of a is the starting value (y-intercept), and b is the growth (multiplier). If $b > 1$, then the function increases (grows); if $0 < b < 1$, then the function decreases (decays). In this course, values of $b < 0$ are not considered.

For additional information, see the Methods & Meanings boxes in Lesson 7.1.4 and 7.2.3. For additional examples and more practice, see Checkpoint 9 and Checkpoint 10A.

Examples

1. LuAnn has \$500 with which to open a savings account. She can open an account at Fredrico's Bank, which pays 7% interest compounded monthly, or Money First Bank, which pays 7.25% compounded quarterly. She plans to leave the money in the account, untouched, for 10 years. In which account should she place the money? Justify your answer.

She should deposit the money into the account that will pay her the most interest over the 10 years.

FREDRICO'S BANK

Fredrico's Bank pays 7% compounded monthly, which means the interest rate is $\frac{0.07}{12} \approx 0.00583$ each month. If LuAnn puts her money into Fredrico's Bank, after 1 month she will have:

$$500 + 500(0.00583) = 500(1.00583) \approx \$502.92$$

To calculate the amount at the end of the 2 months, multiply by 1.00583 again, making the amount:

$$500(1.00583)^2 \approx \$505.85$$

At the end of 3 months, the balance is:

$$500(1.00583)^3 \approx \$508.80$$

This occurs monthly for 120 months (10 years) with the ending balance:

$$500(1.00583)^{120} \approx \$1{,}004.43$$

Note that this last equation is an exponential function in the form $f(x) = ab^x$, where $f(x)$ is the amount of money in the account and x is the number of months (in this case, 120 months). $a = 500$ is the "starting value" (0 months), and $b = 1.00583$ is the multiplier or growth rate for the account each month.

MONEY FIRST BANK

A similar calculation is performed for Money First Bank. Its interest rate is higher, 7.25%, but it is calculated and compounded quarterly. (Quarterly means four times each year.) Hence, every quarter, the bank calculates $\frac{0.0725}{4} = 0.018125$ interest. At the end of the first quarter, LuAnn would have:

$$500(1.018125) \approx \$509.06$$

At the end of 10 years (40 quarters), LuAnn would have:

$$500(1.018125)^{40} \approx \$1{,}025.69$$

Note that this last equation is an exponential function in the form $f(x) = ab^x$, where $f(x)$ is the amount of money in the account and x is the number of quarters (in this case, 40 quarters). $a = 500$ is the starting value (0 quarters), and $b = 1.018125$ is the multiplier or growth rate for the account each quarter.

Since Money First Bank would pay her approximately \$21 more in interest than Fredrico's Bank, she should put her money in Money First Bank.

2. Most homes appreciate in value, at varying rates, depending on the home's location, size, and other factors. But, if a home is used as a rental, the Internal Revenue Service allows the owner to assume that it will depreciate (decrease) in value. Suppose a house that costs $150,000 is used as a rental property and depreciates at a rate of 8% per year. When will the house be worth half of its purchase price?

Use the following questions to guide you to your solution.

- What is the multiplier that will give the value of the house after 1 year?
- What is the value of the house after 1 year?
- What is the value after 10 years?

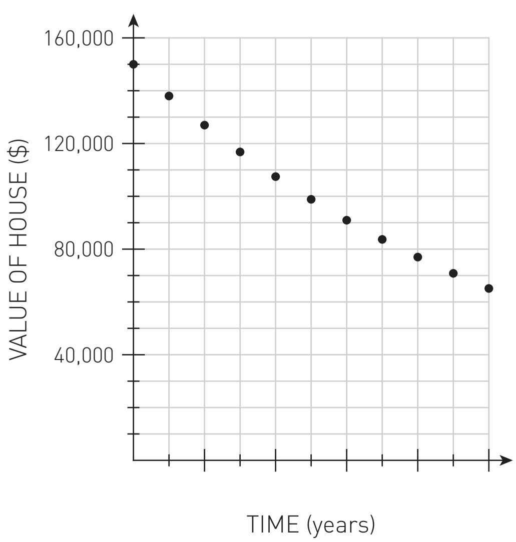
After one year, the value of the house is 150,000 – 0.08(150,000), which is the same as 150,000(0.92). Therefore, the multiplier is 0.92. After one year, the value of the house is 150,000(0.92) = $138,000. After 10 years, the value of the house will be $150{,}000(0.92)^{10}$ = $65,158.27.

This last equation is an exponential function in the form $f(x) = ab^x$, where $f(x)$ is the value of the house and x is the number of years. $a = 150{,}000$ is the starting value (0 years), and $b = 0.92$ is the multiplier or growth rate (in this case, decay rate) each year.

To determine when the house will be worth half of its purchase price, calculate when the house's value reaches $75,000. The value is below $75,000 at 10 years, so this situation occurs in less than 10 years. To help answer this question, list the house's values in a table to see the depreciation.

From the table or graph, you can see that the house will be worth half its purchase price between 8 and 9 years.

Time (years)	**Value of House ($)**
0	150,000
1	138,000
2	126,960
3	116,803.20
4	107,458.94
5	98,862.23
6	90,953.25
7	83,676.99
8	76,982.83
9	70,824.20
10	65,158.27



Note: You can write the equation $75{,}000 = 150{,}000 \cdot 0.92^x$, but you will not have the mathematics to solve this equation for x until a future course. However, you can use the equation to determine a more exact value by trial and error. Try different values for x in the equation, so that $f(x)$ gets closer and closer to $75,000. At about 8.313 years, the house's value is close to $75,000.

3. Write a function that represents the data in the table.

Time (weeks)	Weight of Bacterial Culture (g)
1	756.00
2	793.80
3	833.49

The exponential function will have the form $f(x) = ab^x$, where $f(x)$ is the weight of the bacterial culture and x is the number of weeks. The multiplier, b, for the weight of the bacterial culture is 1.05 (because $793.80 \div 756 = 1.05$ and $833.49 \div 793.80 = 1.05$, etc.). The starting point, a, is not given because the weight at 0 weeks is not given. However, the weight at 0 weeks can be determined by solving $(1.05) \cdot (\text{weight at 0 weeks}) = 756.00$ g. The weight at 0 weeks is 720 g, thus $a = 720$. Then, the function is:

$$f(x) = 720 \cdot 1.05^x$$

Practice

1. In 7 years, Seta's son Stu is leaving home for college. Seta hopes to save $8,000 to help pay for his first year. She has $5,000 now and has found a bank that pays 7.75% interest, compounded daily. At this rate, will she meet her $8,000 goal for Stu's first year of college? If not, how much more does she need?

2. Rudi thought that he was making a sound investment 8 years ago by buying $1,000 worth of Pro Sports Management stock. Unfortunately, his investment depreciated steadily, losing 15% of its value each year. How much is the stock worth after 8 years? Justify your answer.

3. Write an exponential function for each table in the form $f(x) = ab^x$.

a.

x	$f(x)$
1	40
2	32
3	25.6

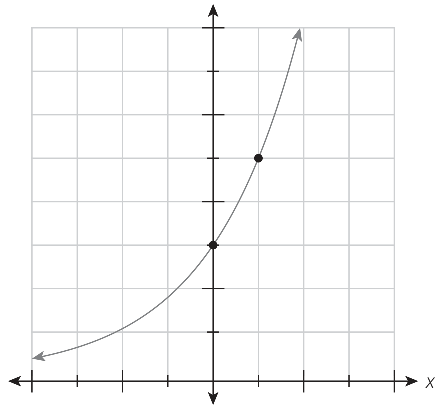
b.

x	$f(x)$
0	1,600
1	2,000
2	2,500
3	3,125

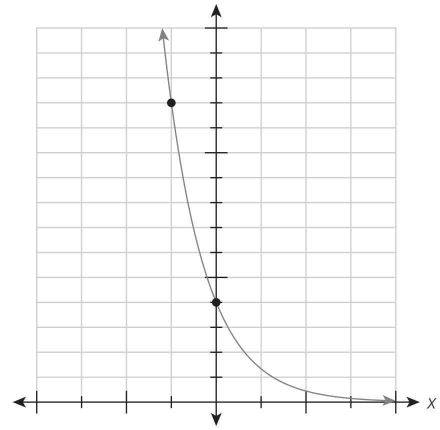
4. The new Bamo Super Ball has a rebound ratio of 0.97. If you drop the ball from a height of 125 feet, how high will it bounce on the tenth bounce?

5. Write an exponential function for each graph in the form $f(x) = ab^x$.

a.



b.



6. Fredrico's Bank will let you decide how often your interest will be computed, but with certain restrictions. If your interest is compounded yearly, you can earn 8%. If your interest is compounded quarterly, you earn 7.875%. Monthly compounding earns a 7.75% interest rate, while weekly compounding earns a 7.625% interest rate. If your interest is compounded daily, you earn 7.5%. Which is the best deal? Justify your answer.

7. Fully investigate the graph of the function $f(x) = \left(\frac{3}{4}\right)^x + 4$. See Describing Graphs of Functions (Lesson 1.2.3) Guided Skill Support for information on how to fully describe the graph of a function.

Answers

1. Yes, she will have about \$8,601.02 in 7 years. The daily rate is $\frac{0.0775}{365} \approx 0.000212329$. There are 2,555 days in 7 years, so $5{,}000(1.000212329)^{2{,}555} \approx \$8{,}601.02$.

2. After 8 years, the stock is worth about \$272.49.

3. a. $f(x) = 1{,}600(1.25)^x$

b. $f(x) = 50(0.8)^x$

4. $125(0.97)^{10} \approx 92.18$ feet

5. a. $f(x) = 3\left(\frac{5}{3}\right)^x$

b. $f(x) = 40\left(\frac{1}{3}\right)^x$

6. One way to solve this problem is to choose an amount and see how it grows over the course of 1 year. Taking \$100, after 1 year:

- 8% compounded yearly will yield \$108.
- 7.875% compounded quarterly yields \$108.11.
- 7.75% compounded monthly yields \$108.03.
- 7.625% compounded weekly yields \$107.91.
- 7.5% compounded daily yields \$107.79.

7.875% compounded quarterly is the best option.

7. $f(x) = \left(\frac{3}{4}\right)^x + 4$ is an exponential function. It has a y-intercept of (0, 5) and no x-intercepts. The domain is all real numbers, and the range is $y > 4$. This function has a horizontal asymptote of $y = 4$ and no vertical asymptotes. The function represents exponential decay because the multiplier, b, is a positive fraction ($0 < b < 1$).

Fractional Exponents

GUIDED SKILL SUPPORT

Overview

A fractional exponent is equivalent to an expression with roots or radicals.

$$\text{For } x \neq 0,\ x^{\frac{a}{b}} = (x^a)^{\frac{1}{b}} = \sqrt[b]{x^a} \text{ or } x^{\frac{a}{b}} = \left(x^{\frac{1}{b}}\right)^a = \sqrt[b]{x^a}.$$

Fractional exponents may also be used to solve equations containing exponents. For additional information, see the Methods & Meanings box in Lesson 7.2.2.

Examples

1. Use exponent properties and roots to evaluate each expression, if possible.

a. $16^{\frac{5}{4}}$

$$16^{\frac{5}{4}} = \left(16^{\frac{1}{4}}\right)^5 = (\sqrt[4]{16})^5 = 2^5 = 32 \text{ or } 16^{\frac{5}{4}} = (16^5)^{\frac{1}{4}} = (1{,}048{,}576)^{\frac{1}{4}} = \sqrt[4]{1{,}048{,}576} = 32$$

b. $(-8)^{\frac{2}{3}}$

$$(-8)^{\frac{2}{3}} = \left(-8^{\frac{1}{3}}\right)^2 = (\sqrt[3]{-8})^2 = (-2)^2 = 4 \text{ or } (-8)^{\frac{2}{3}} = ((-8)^2)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = \sqrt[3]{64} = 4$$

2. Rewrite each expression so that there are no parentheses and no negative exponents.

a. $(144x^{-12})^{\frac{1}{2}}$

$$(144x^{-12})^{\frac{1}{2}} = \left(\frac{144}{x^{12}}\right)^{\frac{1}{2}} = \sqrt{\frac{144}{x^{12}}} = \frac{12}{x^6}$$

b. $\left(\frac{8x^7y^3}{x}\right)^{-\frac{1}{3}}$

$$\left(\frac{8x^7y^3}{x}\right)^{-\frac{1}{3}} = \left(\frac{x}{8x^7y^3}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{1}{8x^6y^3}} = \frac{1}{2x^2y}$$

3. Solve the following equations for x.

a. $x^7 = 42$

Use the property $(x^a)^b = x^{ab}$ to isolate the variable and solve.

$$x^7 = 42$$
$$(x^7)^{\frac{1}{7}} = (42)^{\frac{1}{7}}$$
$$x^{7\left(\frac{1}{7}\right)} = 42^{\frac{1}{7}}$$
$$x^1 = (42)^{\frac{1}{7}}$$
$$x \approx 1.706$$

Note: The final calculation takes the seventh root of 42.

b. $3x^{12} = 132$

$$3x^{12} = 132$$
$$\frac{3x^{12}}{3} = \frac{132}{3}$$
$$x^{12} = 44$$
$$(x^{12})^{\frac{1}{12}} = (44)^{\frac{1}{12}}$$
$$x^1 = (44)^{\frac{1}{12}}$$
$$x \approx \pm 1.371$$

The final calculation takes the twelfth root of 44. Notice that there is only one answer for part (a), where the exponent is odd, but there are two answers (±) here where the exponent is even. Even roots always produce two answers, a positive and a negative. Be sure, if the problem is a real-world application, that both the positive and the negative results make sense before stating both as solutions. You may have to disregard one solution so that the answer is feasible.

Practice

Evaluate the expressions in problems 1 through 3 using properties of exponents and roots.

1. $(64)^{\frac{2}{3}}$

2. $(16)^{-\frac{1}{2}}$

3. $(-27)^{\frac{1}{3}}$

Rewrite the expressions in problems 4 through 6 using the properties of exponents so that there are no negative exponents.

4. $(16a^8b^{12})^{\frac{3}{4}}$

5. $\dfrac{144^{\frac{1}{2}}x^{-3}}{\left(16^{\frac{3}{4}}x^7\right)^0}$

6. $\dfrac{a^{\frac{2}{3}}b^{-\frac{3}{4}}c^{\frac{7}{8}}}{a^{-\frac{1}{3}}b^{\frac{1}{4}}c^{\frac{1}{8}}}$

Solve the equations for x in problems 7 through 12.

7. $x^8 = 65{,}536$

8. $-5x^{-3} = \frac{25}{40}$

9. $3^{5x} = 9^{x-1}$

10. $\left(\frac{2}{3}\right)^x = 1$

11. $2(3x-5)^4 = 512$

12. $2^{4x-1} = 4$

Identify the error in each of the solutions in problems 13 and 14. Then give the correct solutions.

13.
$$\begin{aligned} 4(x+7)^6 &= 1{,}392 \\ (x+7)^6 &= 348 \\ ((x+7)^6)^{\frac{1}{6}} &= (348)^{\frac{1}{6}} \\ x+7 &= 58 \\ x &= 51 \end{aligned}$$

14.
$$\begin{aligned} 54^{x+2} &= 10^{3x-1} \\ 5^{4x+2} &= 2 \cdot 5^{3x-1} \\ 4x+2 &= 2(3x-1) \\ 4x+2 &= 6x-1 \\ 3 &= 2x \\ x &= 1.5 \end{aligned}$$

Answers

1. $(\sqrt[3]{64})^2 = 16$

2. $\frac{1}{\sqrt{16}} = \frac{1}{4}$

3. $\sqrt[3]{-27} = -3$

4. $8a^6b^9$

5. $\frac{12}{x^3}$

6. $\frac{ac^{\frac{3}{4}}}{b}$

7. $x = 4$

8. $x = -2$

9. $x = -\frac{2}{3}$

10. $x = 0$

11. $x = 3$

12. $x = \frac{3}{4}$

13. Both sides need to be raised to the $\frac{1}{6}$ power (or the sixth root taken), not divided by six. $x \approx -4.35$.

14. Since 5 and 10 cannot be written as powers of the same number, the only way to solve the equation now is by guess-and-check. $x \approx 11.75$. There is a second error: the 2 was not distributed in the fourth line.

Curve Fitting Exponential Functions

GUIDED SKILL SUPPORT

Overview

Students write an exponential function of the form $f(x) = ab^x$ that goes through two given points. Equations of this form have an asymptote at $y = 0$.

For additional information, see the Methods & Meanings box in Lesson 9.3.1. For additional examples and more practice, see the Checkpoint 9 and Checkpoint 10A materials.

Examples

1. Determine a possible exponential function that passes through the points $(0, 8)$ and $(4, \frac{1}{2})$.

Substitute the *x*- and *y*-coordinates of each pair of points into the general exponential function $f(x) = ab^x$, where $y = f(x)$. Then solve the resulting system of two equations to determine *a* and *b*.

$(0, 8)$

$$y = ab^x$$
$$8 = ab^0$$
$$8 = a(1)$$
$$8 = a$$

$(4, \frac{1}{2})$

$$y = ab^x$$
$$\frac{1}{2} = ab^4$$

Substitute $a = 8$ into $\frac{1}{2} = ab^4$ and solve for *b*.

$$a = 8$$
$$\frac{1}{2} = ab^4$$
$$\frac{1}{2} = (8)b^4$$
$$\frac{1}{16} = b^4$$
$$\sqrt[4]{\frac{1}{16}} = b^4$$
$$\frac{1}{2} = b$$

Since *a* and *b* have been determined, substitute their values into the function $f(x) = ab^x$ as shown.

$$a = 8 \qquad b = \frac{1}{2}$$
$$f(x) = ab^x$$
$$f(x) = 8\left(\frac{1}{2}\right)^x$$

2. In 2018, Club Leopard formed. By 2022, it had 14,867 members. In 2025, it had risen to 22,610 members. Write an exponential function from this data and use the model to predict the number of members in 2030, assuming this trend continues.

Let x be the number of years since the club formed, and let y be the number of members. Then 2018 is time zero, or $x = 0$, 2022 is $x = 4$, and 2025 is $x = 7$. This gives two data points, (4, 14,867) and (7, 22,610).

To model with an exponential function, substitute both coordinate pairs into $f(x) = ab^x$ to create a system of two equations. Then, rewrite both equations in "$a =$" form in preparation to use the Equal Values Method.

(4, 14,867)

$$y = ab^x$$
$$14{,}867 = ab^4$$
$$\frac{14{,}867}{b^4} = a$$

(7, 22,610)

$$y = ab^x$$
$$22{,}610 = ab^7$$
$$\frac{22{,}610}{b^7} = a$$

Then, by the Equal Values Method:

$$\frac{14{,}867}{b^4} = a \qquad \frac{22{,}610}{b^7} = a$$

$$\frac{14{,}867}{b^4} = \frac{22{,}610}{b^7}$$
$$14{,}867b^7 = 22{,}610b^4$$
$$\frac{b^7}{b^4} = \frac{22{,}610}{14{,}867}$$
$$b^3 = \frac{22{,}610}{14{,}867}$$
$$\sqrt[3]{b^3} = \sqrt[3]{\frac{22{,}610}{14{,}867}}$$
$$b \approx 1.15$$

Solve for a by substituting $b \approx 1.15$ into $a = \frac{14{,}867}{b^4}$ or $a = \frac{22{,}610}{b^7}$ and solving for a.

$$b \approx 1.15$$
$$a = \frac{14{,}867}{b^4}$$
$$a \approx \frac{14{,}867}{(1.15)^4}$$
$$a \approx 8{,}500$$

Since $a \approx 8{,}500$ and $b \approx 1.15$, write the function $f(x) = 8{,}500(1.15)^x$.

To predict the number of members in 2030, calculate $f(12)$.

$$x = 12$$
$$f(x) = 8{,}500(1.15)^x$$
$$f(12) = 8{,}500(1.15)^{12}$$
$$f(12) = 45{,}477$$

Assuming the trend continues, by 2030, there will be approximately 45,477 members.

Practice

Write an exponential function for each pair of points in problems 1 through 4.

1. (0, 6) and (3, 48)

2. (1, 21) and (2, 147)

3. (–1, 72.73) and (3, 106.48)

4. (–2, 351.5625) and (3, 115.2)

5. On a cold winter day, the temperature outside hovered at 0 °C. Karen made herself a cup of cocoa and took it outside, where she would be chopping some wood. Curious about how her cocoa would cool outside, she decided to conduct a mini science experiment. She placed a thermometer in the cocoa and let it sit next to her as she worked. She recorded the time since the thermometer was placed in the cocoa (in minutes) and the temperature (in °C), as shown in the table.

Time (minutes)	5	10	12	15
Temperature (°C)	24.41	8.51	5.58	2.97

Write an exponential function in the form $f(x) = ab^x$ that models this data.

Answers

1. $f(x) = 6(2)^x$

2. $f(x) = 3(7)^x$

3. $f(x) = 80(1.1)^x$

4. $f(x) = 225(0.8)^x$

5. Responses may vary, but should be close to $f(x) = 70(0.81)^x$.

