Factoring Quadratics

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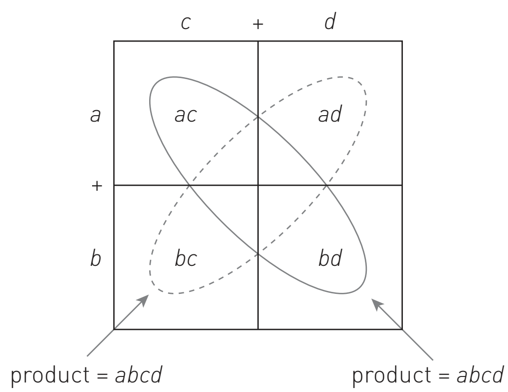
Overview

Quadratic functions are functions that can be rewritten in the form $y = ax^2 + bx + c$ (where $a \neq 0$). When graphed, quadratic functions create a U-shaped curve called a parabola.

There are multiple methods that can be used to factor quadratic equations. One method uses a diagram, a generic rectangle.

Examine the generic rectangle shown for $(a + b)(c + d)$. Notice that each of the resulting diagonals has a product of abcd. Thus, the products of the terms in the diagonals are equal.



The Methods & Meanings boxes in Lessons 8.1.1 through 8.1.4 outline these methods. For additional examples and more practice, see the Checkpoint 10B materials.

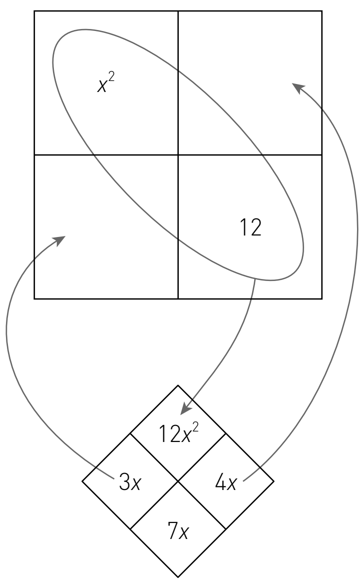
Examples

1. Factor $x^2 + 7x + 12$.

Sketch a generic rectangle and place the x^2 and the 12 along one diagonal.

x^2	
	12

Determine two terms whose product is $12x^2$ and whose sum is $7x$. A Diamond Problem can help determine the factors.



In this case, $3x$ and $4x$. Write these terms along the other diagonal. Either term can go in either diagonal space.

x^2	$4x$
$3x$	12

Determine the greatest common factor of each row and column, then use the base and height of the generic rectangle to determine the areas of the pieces inside the generic rectangle.

	x	+	4
x	x^2		$4x$
+			
3	$3x$		12

Write the sum as a product in factored form.

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

2. Factor $x^2 + 7x - 30$.

Sketch a generic rectangle and place the x^2 and the -30 along one diagonal.

x^2	
	-30

Determine two terms whose product is $-30x^2$ and whose sum is $7x$. In this case, $-3x$ and $10x$. Write these terms along the other diagonal. Either term can go in either diagonal space.

x^2	$10x$
$-3x$	-30

Determine the greatest common factor of each row and column, then use the base and height of the generic rectangle to determine the areas of the pieces inside the generic rectangle.

	x	+ 10
x	x^2	$10x$
+ -3	$-3x$	-30

Write the sum as a product in factored form.

$$x^2 + 7x - 30 = (x - 3)(x + 10)$$

3. Factor $x^2 - 15x + 56$.

Sketch a generic rectangle and place the x^2 and the 56 along one diagonal.

x^2	
	56

Determine two terms whose product is $56x^2$ and whose sum is $-15x$. Write these terms as the other diagonal.

x^2	$-7x$
$-8x$	56

Determine the greatest common factor of each row and column, then use the base and height of the generic rectangle to determine the areas of the pieces inside the generic rectangle.

	x	+	-7
x	x^2		$-7x$
+			
-8	$-8x$		56

Write the sum as a product in factored form.

$$x^2 - 15x + 56 = (x - 8)(x - 7)$$

4. Factor $12x^2 - 19x + 5$.

Sketch a generic rectangle and place the $12x^2$ and the 5 along one diagonal.

Determine two terms whose product is $60x^2$ and whose sum is $-19x$. Write these terms on the other diagonal.

Calculate the base and height of the rectangle. Check the signs of the factors.

$12x^2$	$-4x$
$-15x$	5

	$3x$	$+$	-1
$4x$	$12x^2$		$-4x$
$+$			
-5	$-15x$		5

Write the sum as a product in factored form.

$$(4x - 5)(3x - 1) = 12x^2 - 19x + 5$$

5. Factor $3x^2 + 21x + 36$.

3 is a common factor that appears in all the terms, so it should be factored out first.

$$3x^2 + 21x + 36 = 3(x^2 + 7x + 12)$$

Then $x^2 + 7x + 12$ can be factored in the usual way, as in the other examples.

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

Then, since the expression $3x^2 + 21x + 36$ has a factor of 3.

$$3x^2 + 21x + 36 = 3(x^2 + 7x + 12) = 3(x + 3)(x + 4)$$

Practice

1. $x^2 + 5x + 6$	**2.** $2x^2 + 5x + 3$	**3.** $3x^2 + 4x + 1$	**4.** $3x^2 + 30x + 75$
5. $x^2 + 15x + 44$	**6.** $x^2 + 7x + 6$	**7.** $2x^2 + 22x + 48$	**8.** $x^2 + 4x - 32$
9. $4x^2 + 12x + 9$	**10.** $24x^2 + 22x - 10$	**11.** $x^2 + x - 72$	**12.** $3x^2 - 20x - 7$
13. $x^3 - 11x^2 + 28x$	**14.** $2x^2 + 11x - 6$	**15.** $2x^2 + 5x - 3$	**16.** $x^2 - 3x - 10$
17. $4x^2 - 12x + 9$	**18.** $3x^2 + 2x - 5$	**19.** $6x^2 - x - 2$	**20.** $9x^2 - 18x + 8$

Answers

1. $(x + 2)(x + 3)$	**2.** $(x + 1)(2x + 3)$	**3.** $(3x + 1)(x + 1)$	**4.** $3(x + 5)(x + 5)$
5. $(x + 11)(x + 4)$	**6.** $(x + 6)(x + 1)$	**7.** $2(x + 8)(x + 3)$	**8.** $(x + 8)(x - 4)$
9. $(2x + 3)(2x + 3)$	**10.** $2(3x - 1)(4x + 5)$	**11.** $(x - 8)(x + 9)$	**12.** $(x - 7)(3x + 1)$
13. $x(x - 4)(x - 7)$	**14.** $(x + 6)(2x - 1)$	**15.** $(x + 3)(2x - 1)$	**16.** $(x - 5)(x + 2)$
17. $(2x - 3)(2x - 3)$	**18.** $(3x + 5)(x - 1)$	**19.** $(2x + 1)(3x - 2)$	**20.** $(3x - 4)(3x - 2)$

Factoring Strategies

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Overview

Although most factoring problems can be done with generic rectangles, there are two special factoring patterns that, if recognized, can be done by sight. The two patterns are known as the **Difference of Squares** and **Perfect Square Trinomials**. The general patterns are as follows:

Difference of Squares: $a^2x^2 - b^2y^2 = (ax + by)(ax - by)$

Perfect Square Trinomial: $a^2x^2 + 2abxy + b^2y^2 = (ax + by)^2$

Examples

1. Factor $x^2 - 49$

Use the difference of squares.

$$x^2 - 49 = (x + 7)(x - 7)$$

2. Factor $4x^2 - 25$

Use the difference of squares.

$$4x^2 - 25 = (2x - 5)(2x + 5)$$

3. Factor $x^2 - 10x + 25$

Use the perfect square trinomial.

$$x^2 - 10x + 25 = (x - 5)^2$$

4. Factor $9x^2 + 12x + 4$

Use the perfect square trinomial.

$$9x^2 + 12x + 4 = (3x + 2)^2$$

5. Factor $8x^2 + 50y^2$

Sometimes removing a common factor reveals one of the special patterns:

$$8x^2 - 50y^2 = 2(4x^2 - 25y^2) = 2(2x + 5y)(2x - 5y)$$

Practice

Factor each difference of squares.

1. $x^2 - 16$

2. $x^2 - 25$

3. $64m^2 - 25$

4. $4p^2 - 9q^2$

5. $9x^2y^2 - 49$

6. $x^4 - 25$

7. $64 - y^2$

8. $144 - 25p^2$

9. $9x^4 - 4y^2$

Factor each perfect square trinomial.

10. $x^2 + 4x + 4$

11. $y^2 + 8y + 16$

12. $m^2 - 10m + 25$

13. $x^2 - 8x + 16$

14. $a^2 + 8ab + 16b^2$

15. $36x^2 + 12x + 1$

16. $25x^2 - 30xy + 9y^2$

17. $9x^2y^2 - 6xy + 1$

18. $49x^2 + 1 + 14x$

Factor completely.

19. $9x^2 - 16$

20. $9x^2 + 24x + 16$

21. $9x^2 - 36$

22. $2x^2 + 8xy + 8y^2$

23. $x^2y + 10xy + 25y$

24. $8x^2 - 72$

25. $4x^3 - 9x$

26. $4x^2 - 8x + 4$

27. $2x^2 + 8$

Answers

1.	$(x + 4)(x - 4)$	**2.**	$(x + 5)(x - 5)$	**3.**	$(8m + 5)(8m - 5)$
4.	$(2p + 3q)(2p - 3q)$	**5.**	$(3xy + 7)(3xy - 7)$	**6.**	$(x^2 + 5)(x^2 - 5)$
7.	$(8 + y)(8 - y)$	**8.**	$(12 + 5p)(12 - 5p)$	**9.**	$(3x^2 + 2y)(3x^2 - 2y)$
10.	$(x + 2)^2$	**11.**	$(y + 4)^2$	**12.**	$(m - 5)^2$
13.	$(x - 4)^2$	**14.**	$(a + 4b)^2$	**15.**	$(6x + 1)^2$
16.	$(5x - 3y)^2$	**17.**	$(3xy - 1)^2$	**18.**	$(7x + 1)^2$
19.	$(3x + 4)(3x - 4)$	**20.**	$(3x + 4)^2$	**21.**	$9(x + 2)(x - 2)$
22.	$2(x + 2y)^2$	**23.**	$y(x + 5)^2$	**24.**	$8(x + 3)(x - 3)$
25.	$x(2x + 3)(2x - 3)$	**26.**	$4(x - 1)^2$	**27.**	$2(x^2 + 4)$

Using the Zero Product Property

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Overview

Students will learn multiple methods to solve quadratic equations in this chapter and in Chapter 9. One method to solve quadratic equations uses the Zero Product Property, that is, solving by factoring. This method uses two ideas:

1. When the product of two or more numbers is zero, then one of the numbers must be zero.
2. Some quadratic expressions can be factored into the product of two binomials.

For additional information, see the Methods & Meanings box in Lesson 8.2.2.

Examples

1. Determine the x-intercepts of $y = x^2 + 6x + 8$.

$y = x^2 + 6x + 8$

$\downarrow$

$0 = x^2 + 6x + 8$

The x-intercepts are located on the graph where $y = 0$, so write the quadratic expression equal to zero.

$0 = (x + 4)(x + 2)$

Factor the quadratic expression.

$(x + 4) = 0$ $\quad$ $(x + 2) = 0$

Set each factor equal to 0.

$x = -4$ $\quad$ $x = -2$

Solve each equation for x.

The x-intercepts are $(-4, 0)$ and $(-2, 0)$.

You can check your answers by substituting them into the original equation.

$x = \boxed{-4}$

$$\begin{aligned} x^2 + 6x + 8 &= 0 \\ (-4)^2 + 6(-4) + 8 &= 0 \\ 16 - 24 + 8 &= 0 \\ 0 &= 0 \end{aligned}$$

true

$x = \boxed{-2}$

$$\begin{aligned} x^2 + 6x + 8 &= 0 \\ (-2)^2 + 6(-2) + 8 &= 0 \\ 4 - 12 + 8 &= 0 \\ 0 &= 0 \end{aligned}$$

true

2. Solve $2x^2 + 7x - 15 = 0$.

$2x^2 + 7x - 15 = 0$

$(2x - 3)(x + 5) = 0$ Factor the quadratic expression.

$(2x - 3) = 0$ $(x + 5) = 0$ Set each factor equal to 0.

$2x = 3$ $x = -5$ Solve each equation for x.

$x = \frac{3}{2}$

3. Solve $2 = 6x^2 - x$.

If the quadratic equation does not equal 0, rewrite it algebraically so that it does, then use the Zero Product Property.

$2 = 6x^2 - x$

$0 = 6x^2 - x - 2$

$0 = (2x + 1)(3x - 2)$ Factor the quadratic expression.

$(2x + 1) = 0$ $(3x - 2) = 0$ Set each factor equal to 0.

$2x = -1$ $3x = 2$ Solve each equation for x.

$x = -\frac{1}{2}$ $x = \frac{2}{3}$

4. $9x^2 - 6x + 1 = 0$.

Factor the quadratic expression.

$9x^2 - 6x + 1 = 0$

$\downarrow$

$(3x - 1)(3x - 1) = 0$

$\downarrow$

$(3x - 1) = 0$

$x = \frac{1}{3}$

Practice

Solve for x.

1. $x^2 - x - 12 = 0$

2. $3x^2 - 7x - 6 = 0$

3. $x^2 + x - 20 = 0$

4. $3x^2 + 11x + 10 = 0$

5. $x^2 + 5x = -4$

6. $6x - 9 = x^2$

7. $6x^2 + 5x - 4 = 0$

8. $x^2 - 6x + 8 = 0$

9. $6x^2 - x - 15 = 0$

10. $4x^2 + 12x + 9 = 0$

11. $x^2 - 12x = 28$

12. $2x^2 + 8x + 6 = 0$

13. $2 + 9x = 5x^2$

14. $2x^2 - 5x = 3$

15. $x^2 = 45 - 4x$

Answers

1. $x = 4$ or -3

2. $x = -\frac{2}{3}$ or 3

3. $x = -5$ or 4

4. $x = -\frac{5}{3}$ or -2

5. $x = -4$ or -1

6. $x = 3$

7. $x = -\frac{4}{3}$ or $\frac{1}{2}$

8. $x = 4$ or 2

9. $x = -\frac{3}{2}$ or $\frac{5}{3}$

10. $x = -\frac{3}{2}$

11. $x = 14$ or -2

12. $x = -1$ or -3

13. $x = -\frac{1}{5}$ or 2

14. $x = -\frac{1}{2}$ or 3

15. $x = 5$ or -9

Graphing Form and Completing the Square

GUIDED SKILL SUPPORT

Overview

In Lesson 8.2.3, students learned that if the equation of a parabola is written in **graphing form**, $f(x) = (x - h)^2 + k$, then the vertex can easily be seen as (h, k). For example, for the parabola $f(x) = (x + 3)^2 - 1$, the vertex is $(-3, -1)$. Students can then set the function equal to zero to calculate the x-intercepts: $0 = (x + 3)^2 - 1$. Students can set $x = 0$ to determine the y-intercepts: $y = (0 + 3)^2 - 1$.

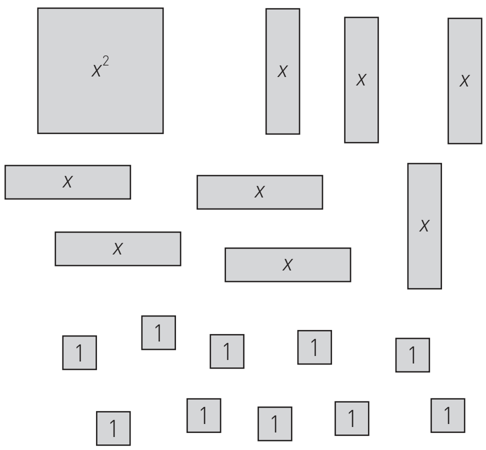
When the equation of the parabola is given in standard form, $f(x) = x^2 + bx + c$, then the process of **completing the square** can be used to convert standard form into graphing form. Algebra tiles can help visualize the process.

For additional examples and practice with graphing quadratic functions, see the Checkpoint 11 materials at the back of the student textbook.

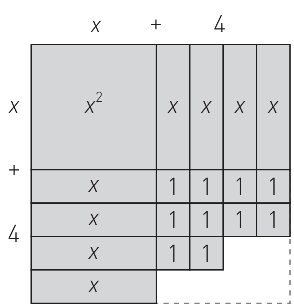
Examples

1. Complete the square to change $f(x) = x^2 + 8x + 10$ into graphing form, identify the vertex and y-intercept, and draw a graph.

Use algebra tiles. $f(x) = x^2 + 8x + 10$ would look like this:



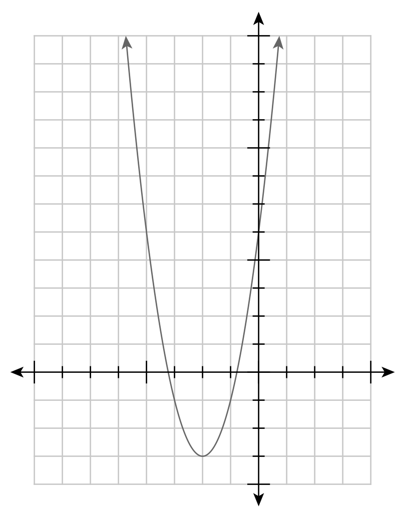
Arrange the tiles as shown to make a square. 16 small unit tiles are needed to fill in the corner, but only 10 unit tiles are available. Show the 10 small square tiles and draw the outline of the whole square.



The **complete square** would have length and width both equal to $(x + 4)$, so the complete square can be represented by the quadratic expression $(x + 4)^2$. But the tiles from $x^2 + 8x + 10$ do not form a complete square—the expression $x^2 + 8x + 10$ has six fewer unit tiles than a complete square. So $x^2 + 8x + 10$ is a complete square minus 6, or $(x + 4)^2$ minus 6. That is:

$$x^2 + 8x + 10 = (x + 4)^2 - 6$$

From graphing form, the vertex is at $(h, k) = (-4, -6)$. The y-intercept is where $x = 0$, so $y = (0 + 4)^2 - 6 = 10$. The y-intercept is $(0, 10)$, and the graph is shown.



2. Complete the square to change $f(x) = x^2 + 5x + 2$ into graphing form. Identify the vertex and y-intercept, and draw a graph.

Draw a generic rectangle of $x^2 + 5x + 2$ and imagine algebra tiles.

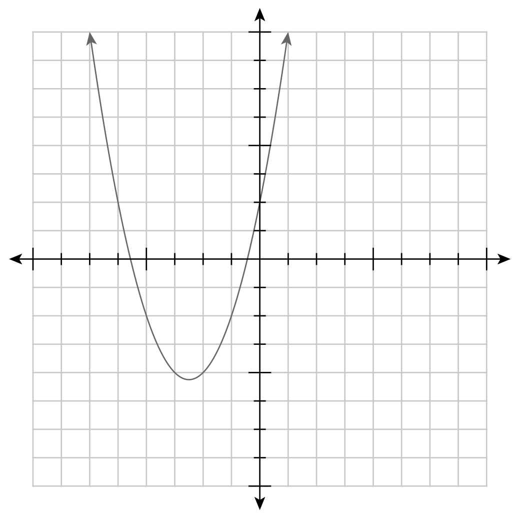
x^2	$2.5x$
$2.5x$	

There should be $(-2.5)^2 = 6.25$ tiles in the lower right corner to complete the square. But the expression $x^2 + 5x + 2$ only provides 2 unit tiles. So there are 4.25 unit tiles missing. Thus, 4.25 is missing from the rectangle shown:

	x +	2.5
x +	x^2	$2.5x$
2.5	$2.5x$	

That is, $(x + 2.5)^2$ minus 4.25 unit tiles is the equivalent of $x^2 + 5x + 2$, or,
$x^2 + 5x + 2 = (x + 2.5)^2 - 4.25$.

The function is now in graphing form.
The vertex is $(-\frac{5}{2}, -\frac{17}{4})$ or $(-2.5, -4.25)$.
The y-intercept is where $x = 0$. Thus,
$y = (0 + 2.5)^2 - 4.25 = 2$ and the y-intercept is $(0, 2)$.



Practice

Complete the square to write each equation in graphing form. Then state the vertex.

1. $f(x) = x^2 + 6x + 7$

2. $f(x) = x^2 + 4x + 11$

3. $f(x) = x^2 + 10x$

4. $f(x) = x^2 + 7x + 2$

5. $f(x) = x^2 - 6x + 9$

6. $f(x) = x^2 + 3$

7. $f(x) = x^2 - 4x$

8. $f(x) = x^2 + 2x - 3$

9. $f(x) = x^2 + 5x + 1$

10. $f(x) = x^2 - \frac{1}{3}x$

Answers

1. $f(x) = (x + 3)^2 - 2$; $(-3, -2)$

2. $f(x) = (x + 2)^2 + 7$; $(-2, 7)$

3. $f(x) = (x + 5)^2 - 25$; $(-5, -25)$

4. $f(x) = (x + 3.5)^2 - 10.25$; $(-3.5, -10.25)$

5. $f(x) = (x - 3)^2$; $(3, 0)$

6. $f(x) = x^2 + 3$; $(0, 3)$

7. $f(x) = (x - 2)^2 - 4$; $(2, -4)$

8. $f(x) = (x + 1)^2 - 4$; $(-1, -4)$

9. $f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{21}{4}$; $(-\frac{5}{2}, -\frac{21}{4})$

10. $f(x) = \left(x - \frac{1}{6}\right)^2 - \frac{1}{36}$; $(\frac{1}{6}, -\frac{1}{36})$