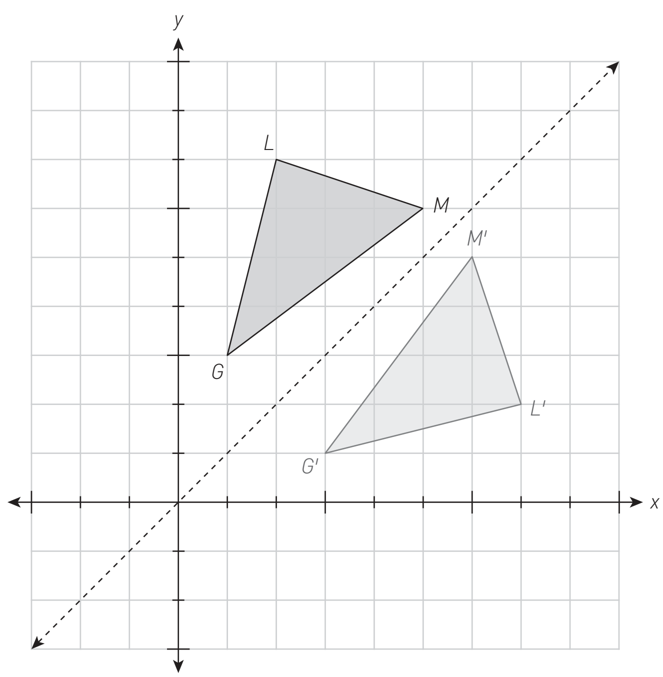
Transformations and Symmetry

GUIDED SKILL SUPPORT

Overview

The study of rigid transformations of geometric shapes builds a foundation for the geometric concept of congruence. When applying the three rigid transformations (translations, reflections, and rotations) to a preimage, the image of the figure moves to a new position without changing its size or shape. Prime marks are used to indicate the new locations of each point after a transformation on a coordinate plane.



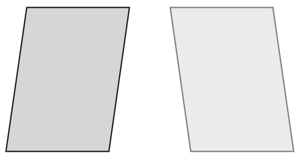
In addition to congruence, rigid transformations build an understanding of symmetry in shapes. Symmetry can be used to describe and classify geometric shapes. In this course, students explore which polygons have lines of symmetry and which have rotational symmetry.

Additional information about rigid transformations can be found in the Lesson 3.1.5 Methods & Meanings box.

Examples

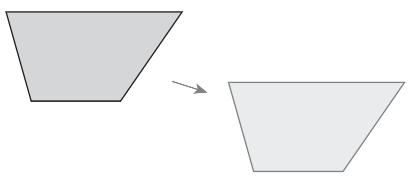
1. Decide which rigid transformation(s) were used on each pair of shapes.

a.



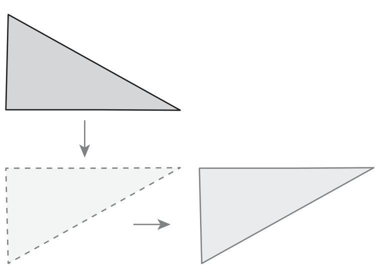
The parallelogram is reflected (flipped) across an invisible vertical line of reflection. Reflecting a shape once changes its orientation. For example, the two sides of the left figure slant upwards to the right, whereas the sides of its reflection slant upwards to the left. Likewise, the angles in the left figure (preimage) are not in the same position as the angles in its image. For example, the obtuse angle in the top left corner of the preimage is located at the top right corner of the image.

b.



The shape is translated (slid) to the right and down. The orientation remains the same, with all corresponding angles and sides slanting the same.

c.

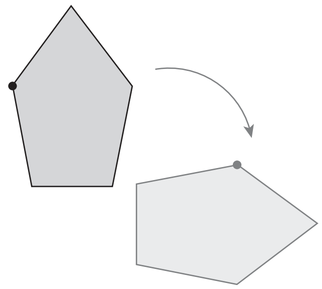


This is a reflection followed by a translation. First, the triangle is flipped across an invisible horizontal line of reflection. As a result, the orientation of the figure changed. Then the figure is translated to the right. The final image has the same orientation as the previous image.

Example continues on next page.

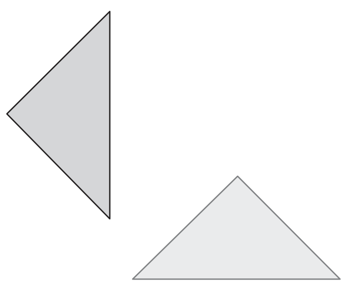
1. Example continued from previous page.

d.



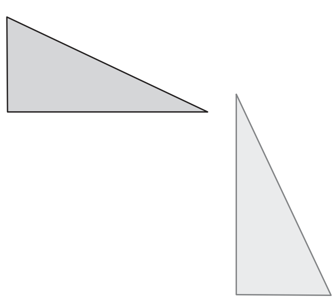
The pentagon is rotated (turned) 90° clockwise to create the second figure. Imagine tracing the first figure on tracing paper, holding the tracing paper with a pin at one point below the first pentagon, then turning the paper to the right by 90°. The second pentagon would be the result. It might seem as though this is a reflection across a diagonal line. The pentagon itself could be reflected, but the movement of the dot indicates the figure was not reflected. If the pentagon was reflected, the dot would be on the corner directly below the one shown in the rotated image.

e.



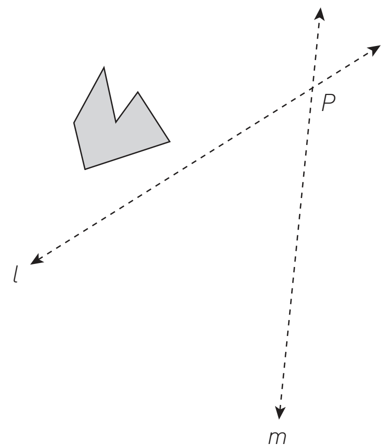
The triangle is rotated 90° clockwise. The image has a different orientation than the original triangle.

f.

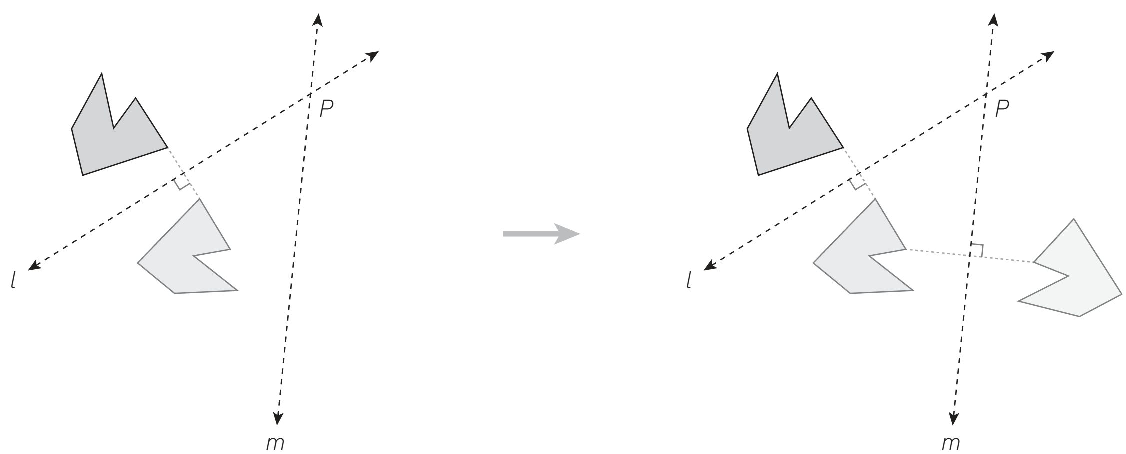


Although both transformations are not shown, the triangle is rotated, then reflected. First, the triangle is rotated 90° counterclockwise. This is evident by the shortest side becoming horizontal instead of vertical. Then, the triangle is reflected across a vertical line of reflection. The orientation of the original triangle is changed twice as result.

2. The hexagon shown will be reflected across line l, and then the image will be reflected across line m. How does the final hexagon compare to the original hexagon? Use tracing paper to support your answer.

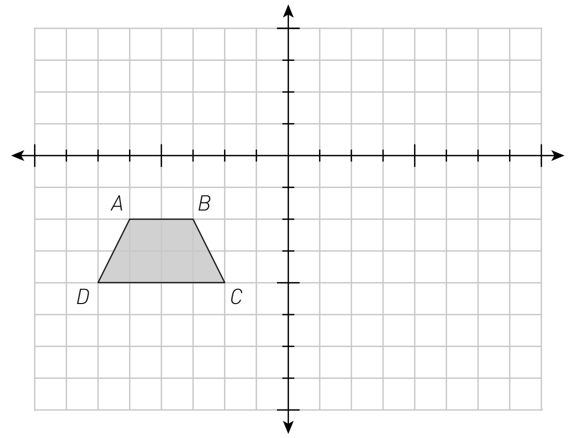


The first reflection across line l positions the image between lines l and m. If you draw a line between corresponding vertices, this line is perpendicular to line l. Also, the corresponding vertices are the same distance from line l. The orientation changes after reflection. Use tracing paper to draw the reflection by tracing the figure and line l. Then turn the tracing paper over, so that line l is on top of itself. This reveals the position of the reflection. Transfer the figure to the paper by tracing it. Repeat this process with line m to determine the location of the hexagon after a reflection across line m.



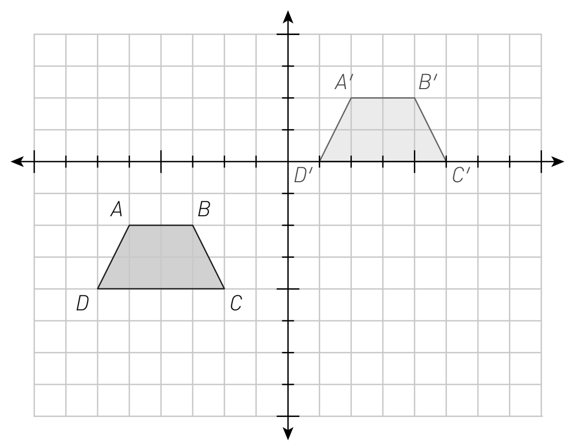
Reflecting twice across two intersecting lines produces a rotation of the original figure about the point P. To visualize this movement, you can use tracing paper. Put the tracing paper back in the original location of the hexagon. Place the point of a pen or pencil on the tracing paper at point P (the intersection of the lines of reflection) and rotate the tracing paper until the original figure aligns with the last image.

3. Consider trapezoid *ABCD* shown.



a. Translate the trapezoid *ABCD* 7 units to the right and 4 units up. Label the new trapezoid *A′B′C′D′* and give the coordinates of its vertices.

Translating (sliding) the trapezoid gives a new trapezoid with vertices at *A′*(2, 2), *B′*(4, 2), *C′*(5, 0), and *D′*(1, 0).



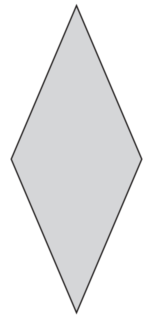
b. Is it possible to translate the original trapezoid in such a way as to create *A″B″C″D″* so that it is a reflection of *ABCD*? If so, what would be the line of reflection?

Since the trapezoid *ABCD* is symmetrical, it is possible to translate it in such a way that makes it look as if it were a reflection rather than a translation. The trapezoid can be slid horizontally left or right. In either case, the resulting figure would look like a reflection across a vertical line.

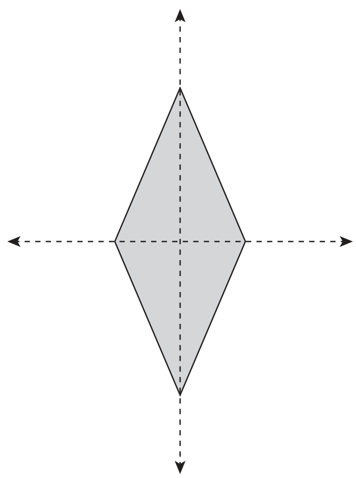
c. Is it always possible to translate a preimage so that the image is also a reflection?

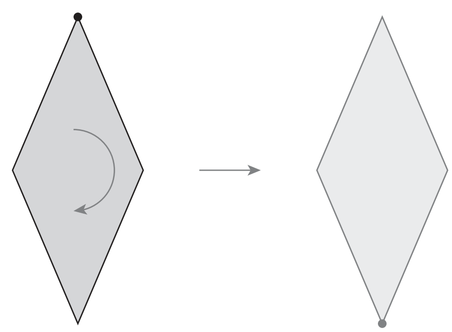
This will not always work. It works for the isosceles trapezoid example because the preimage has a line of symmetry.

4. Draw all lines of symmetry and describe the rotational symmetry.



Lines of symmetry divide the shape into identical halves. This figure has a vertical and a horizontal line of symmetry. This figure has 180° rotational symmetry, because if it is rotated 180°, it is in alignment with the original figure.

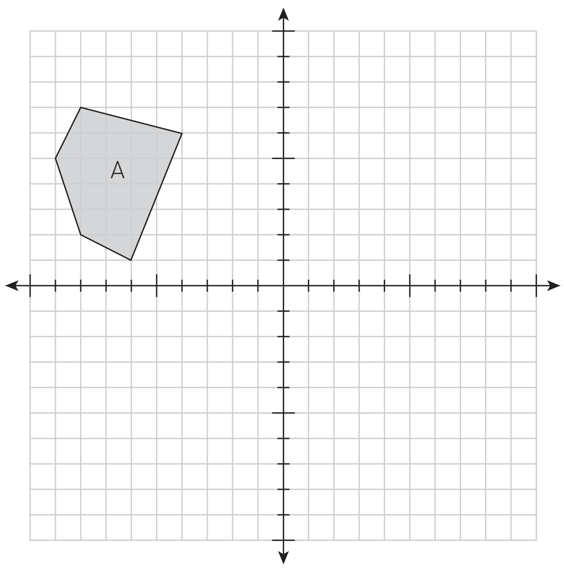




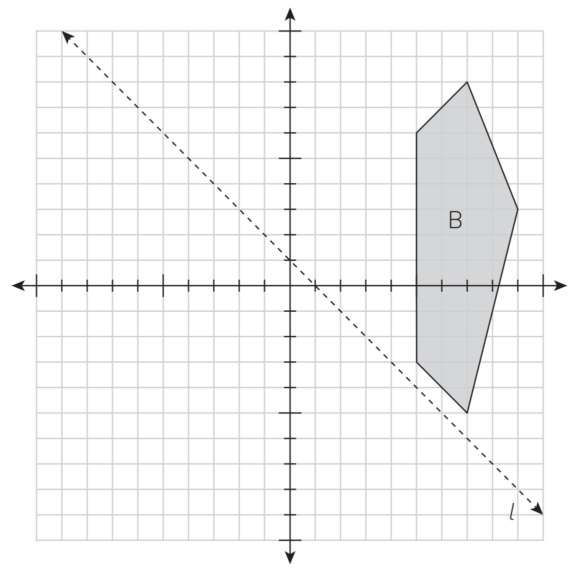
Practice

Perform the indicated transformation on each polygon. Draw the image after the transformation. Use tracing paper to see how the figure moves.

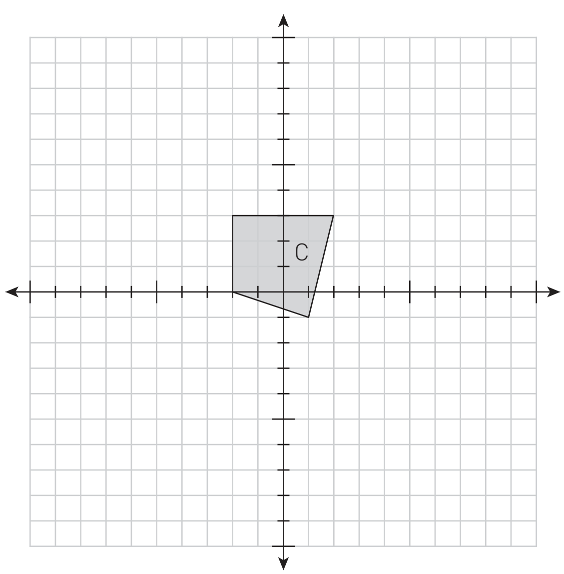
1. Rotate Figure A 90° clockwise about the origin.



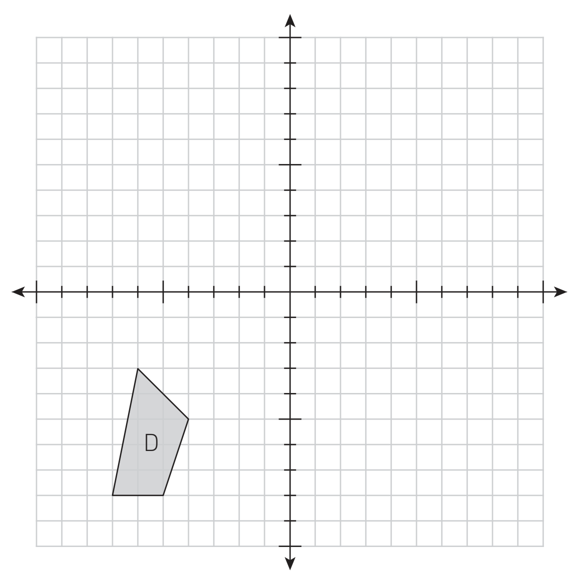
2. Reflect Figure B across line *l*.



3. Translate Figure C 6 units to the left.



4. Rotate Figure D 270° clockwise about the origin (0, 0).



For problems 5 through 20, refer to the figures shown. What are the new coordinates of *A*, *B*, and *C* after each transformation?

FIGURE 1

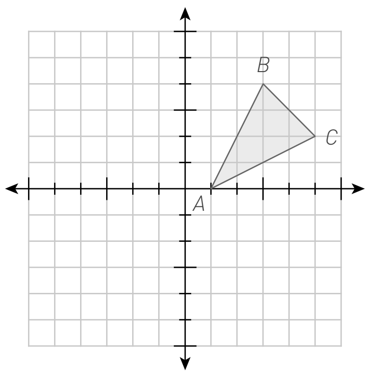


FIGURE 2

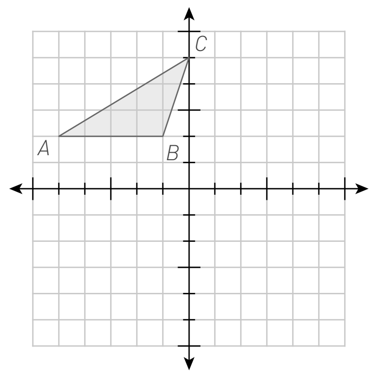
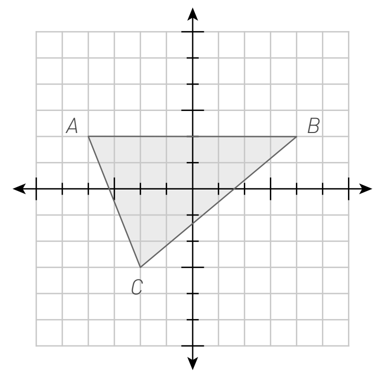


FIGURE 3



5. Translate the triangle in Figure 1 left 2 units and down 3 units.

6. Translate the triangle in Figure 2 right 3 units and down 5 units.

7. Translate the triangle in Figure 3 left 1 unit and up 2 units.

8. Reflect the triangle in Figure 1 across the *x*-axis.

9. Reflect the triangle in Figure 2 across the *x*-axis.

10. Reflect the triangle in Figure 3 across the *x*-axis.

11. Reflect the triangle in Figure 1 across the *y*-axis.

12. Reflect the triangle in Figure 2 across the *y*-axis.

13. Reflect the triangle in Figure 3 across the *y*-axis.

14. Rotate the triangle in Figure 1 90° counterclockwise about the origin.

15. Rotate the triangle in Figure 2 90° counterclockwise about the origin.

16. Rotate the triangle in Figure 3 90° counterclockwise about the origin.

17. Rotate the triangle in Figure 1 180° counterclockwise about the origin.

18. Rotate the triangle in Figure 3 180° counterclockwise about the origin.

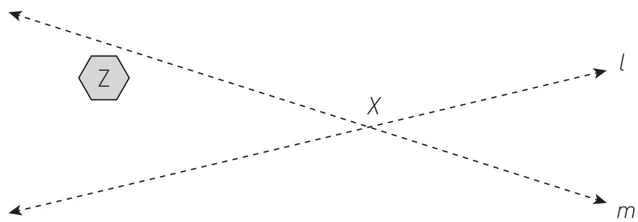
19. Rotate the triangle in Figure 2 270° counterclockwise about the origin.

20. Rotate the triangle in Figure 3 90° clockwise about the origin.

21. Plot the points $A(3, 3)$, $B(6, 1)$, and $C(3, -4)$. Translate the triangle 8 units to the left and 1 unit up to create $\Delta A'B'C'$. What are the coordinates of the vertices of the new triangle?

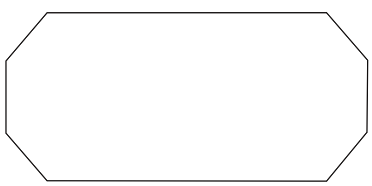
22. What translation moves A from ΔABC in problem 21 to point $A''(4, -5)$?

23. Reflect Figure Z across line l, and then reflect the new figure across line m. What other transformation are these two reflections equivalent to?



For each shape, draw all lines of symmetry and describe its rotational symmetry if it has any.

24.



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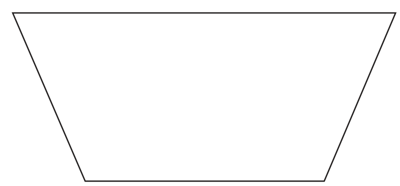

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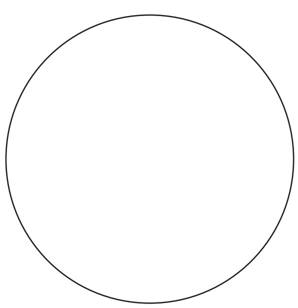

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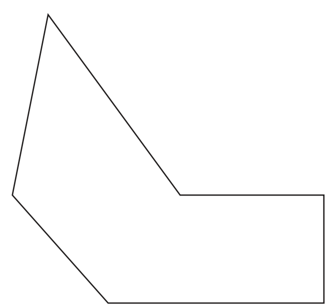
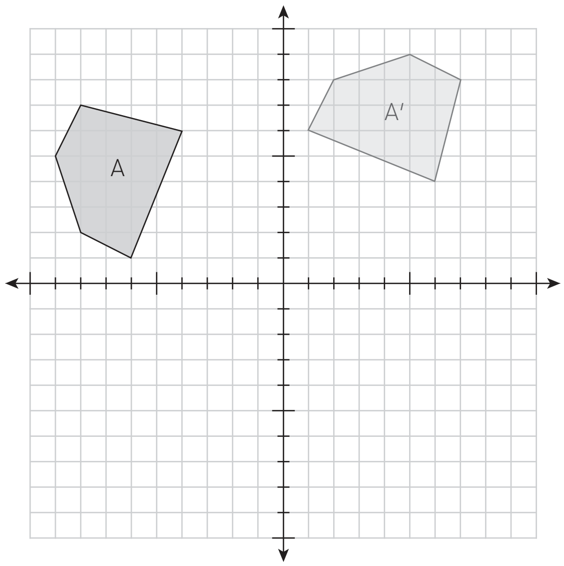
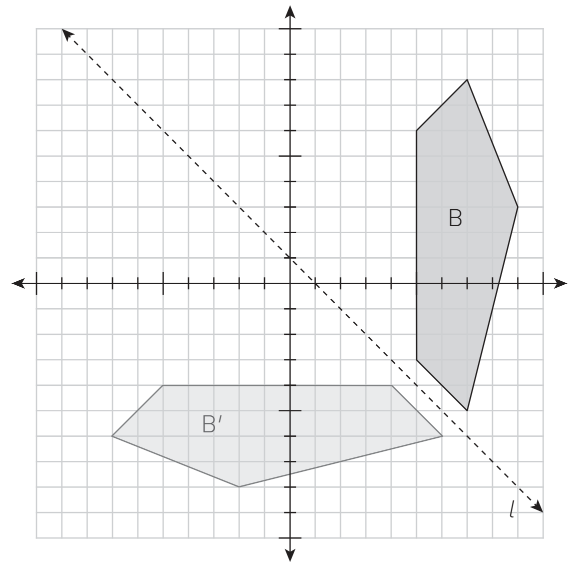


Answers

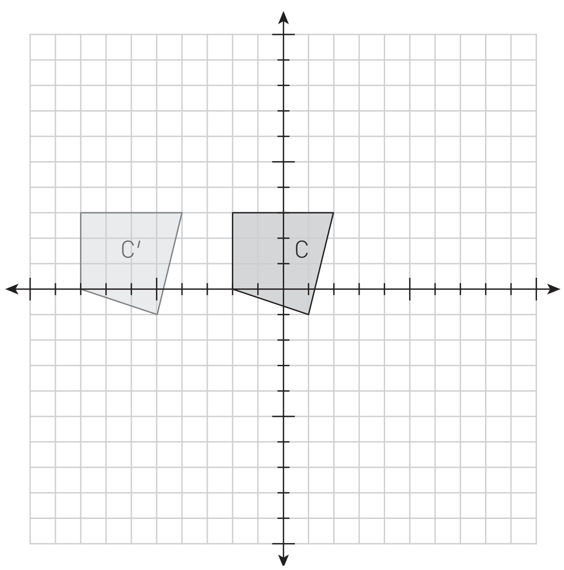
1.



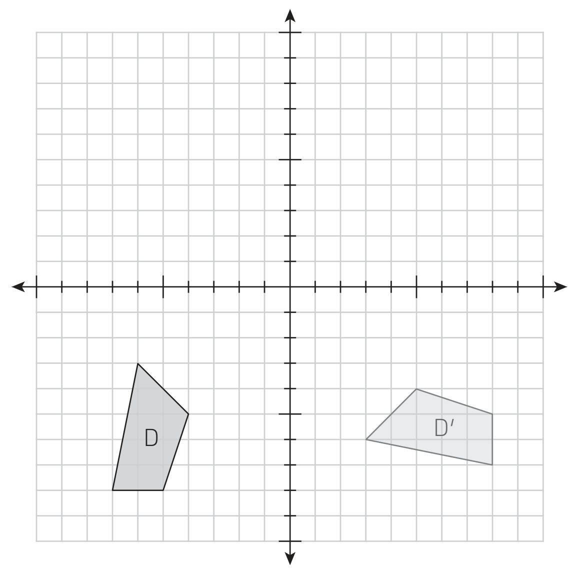
2.



3.



4.



5. $A(-1, -3)$, $B(1, 1)$, $C(3, -1)$

6. $A(-2, -3)$, $B(2, -3)$, $C(3, 0)$

7. $A(-5, 4)$, $B(3, 4)$, $C(-3, -1)$

8. $A(1, 0)$, $B(3, -4)$, $C(5, -2)$

9. $A(-5, -2)$, $B(-1, -2)$, $C(0, -5)$

10. $A(-4, -2)$, $B(4, -2)$, $C(-2, 3)$

11. $A(-1, 0)$, $B(-3, 4)$, $C(-5, 2)$

12. $A(5, 2)$, $B(1, 2)$, $C(0, 5)$

13. $A(4, 2)$, $B(-4, 2)$, $C(2, -3)$

14. $A(0, 1)$, $B(-4, 3)$, $C(-2, 5)$

15. $A(-2, -5)$, $B(-2, -1)$, $C(-5, 0)$

16. $A(-2, -4)$, $B(-2, 4)$, $C(3, -2)$

17. $A(-1, 0)$, $B(-3, -4)$, $C(-5, -2)$

18. $A(4, -2)$, $B(-4, -2)$, $C(2, 3)$

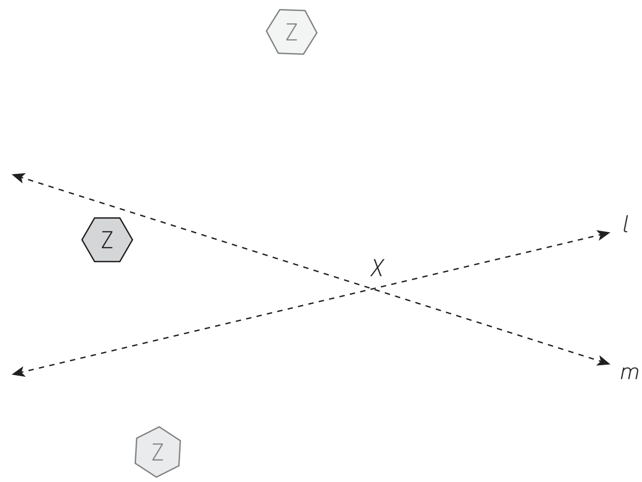
19. $A(2, 5)$, $B(2, 1)$, $C(5, 0)$

20. $A(2, 4)$, $B(2, -4)$, $C(-3, 2)$

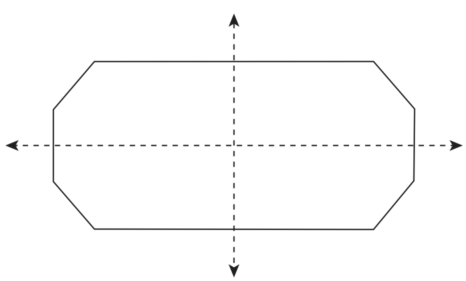
21. $A'(-5, 4)$, $B'(-2, 2)$, $C'(-5, -3)$

22. Translate it 1 unit right and 8 units down.

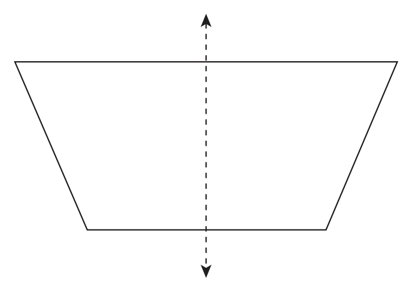
23. The two reflections are the same as rotating Z about point *X*.



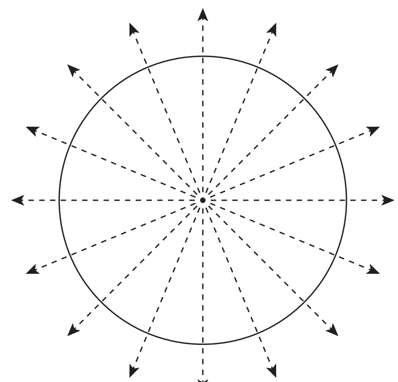
24. This has 180° rotational symmetry.



25. No rotational symmetry.



26. A circle has infinitely many lines of symmetry. A few examples are provided. It has all possible degrees of rotational symmetry.



27. No lines of symmetry and no rotational symmetry.

Slopes of Parallel and Perpendicular Lines

GUIDED SKILL SUPPORT

Overview

The most commonly used form of the equation of a line is slope–intercept form or $y = mx + b$, where m is the slope and b is the y-intercept. Parallel lines have the same slope. For perpendicular lines, the product of the slopes equals –1.

Additional information about the slopes of parallel and perpendicular lines can be found in the Lesson 3.1.7 Methods & Meanings box.

Examples

Using slope–intercept form, write an equation that represents each line described.

1. The line that is parallel to $y = \frac{1}{2}x - 5$ and passes through the point (4, 10).

Any line parallel to the line $y = \frac{1}{2}x - 5$ also has a slope of $\frac{1}{2}$.

$m = \frac{1}{2}$ $x = 4$ $y = 10$

$y = mx + b$ Substitute (4, 10) for x and y.

$10 = \left(\frac{1}{2}\right)(4) + b$ Solve for b.

$10 = 2 + b$

$8 = b$

$m = \frac{1}{2}$

$y = mx + b$ Use the value of b to write the complete equation.

$y = \frac{1}{2}x + 8$

2. The line that is perpendicular to $y = \frac{1}{2}x - 5$ and passes through the point (4, 10).

Since the original line has a slope of $\frac{1}{2}$, the slope of any perpendicular line must be −2 so that the product of the slopes equals −1.

$m = -2 \quad x = 4 \quad y = 10$

$y = mx + b$

$10 = (-2)(4) + b$ Substitute (4, 10) for x and y.

$10 = -8 + b$ Solve for b.

$18 = b$

$m = -2$

$y = mx + b$ Use the value of b to write the complete equation.

$y = -2x + 18$

Practice

Write an equation for each of the lines described in slope–intercept form, when possible.

1. The line with a slope of $\frac{1}{3}$, passing through the point (−2, 5).

2. The line parallel to $y = \frac{2}{3}x + 5$, passing through the point (3, 2).

3. The line parallel to $y = -\frac{3}{4}x - 2$, passing through the point (−4, 4).

4. The line parallel to the line containing the points (−3, −2) and (2, 4), passing through the point (0, −1).

5. The line parallel to $y = 7$, passing through the point (−2, 5).

6. The line through the point (0, 0) and parallel to $y = 2x - 3$.

7. The line perpendicular to $y = \frac{2}{3}x + 5$, passing through the point (3, 2).

8. The line perpendicular to $y = -\frac{3}{4}x + 1$, passing through the point (−4, 2).

9. The line perpendicular to $y = \frac{2}{3}x - 2$, passing through the point (0, 3).

10. The line perpendicular to $y = 7$, passing through the point (−2, 5).

Answers

1. $y = \frac{1}{3}x + 5\frac{2}{3}$

2. $y = \frac{2}{3}x$

3. $y = -\frac{3}{4}x + 1$

4. $y = \frac{6}{5}x - 1$

5. $y = 5$

6. $y = 2x$

7. $y = -\frac{3}{2}x + 6\frac{1}{2}$

8. $y = \frac{4}{3}x + \frac{22}{3}$

9. $y = -\frac{3}{2}x + 3$

10. $x = -2$

Multiplying Binomials

GUIDED SKILL SUPPORT

Overview

Two ways to determine the area of a rectangle are:

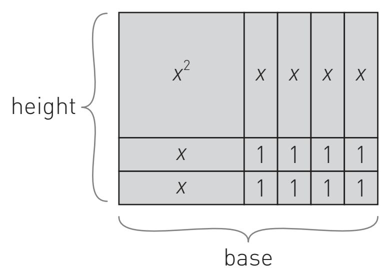
- as a product of the height and base: (height) · (base); or
- as the sum of the areas of individual pieces of the rectangle: $\text{area}_1 + \text{area}_2 + \ldots\text{area}_n$.

For all rectangles, the **area as a product is equal to the area as a sum**. Algebra tiles and generic rectangles can represent area models used to visually represent multiplying expressions.

Additional information about multiplying binomials can be found in the Lessons 3.2.1 and 3.2.2 Methods & Meanings boxes. For additional examples and practice, refer to Checkpoint 6B materials.

Examples

1. Write an equation showing the area of the rectangle as a product and as a sum.



AREA AS A PRODUCT

Determine the base and height of the rectangle by combining algebra tiles. The height is $x + 2$ and the base is $x + 4$. Multiply the base and height to represent the area of the algebra tiles, $(x + 4)(x + 2)$.

AREA AS A SUM

Add the areas of the algebra tiles, $x^2 + x + x + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + x + x + x + x$. Rewrite the expression by combining like terms, $x^2 + 6x + 8$.

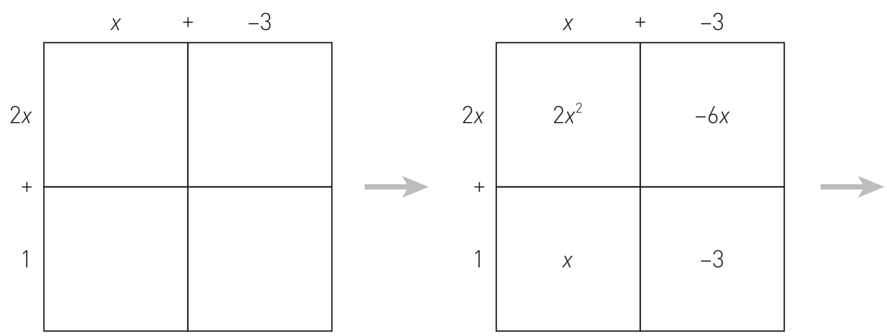
Write an equation showing that both representations of the area are equal.

$$\underbrace{(x + 4)}_{\text{base}}\underbrace{(x + 2)}_{\text{height}} = \underbrace{x^2 + 6x + 8}_{\text{area}}$$

2. Use a generic rectangle to determine the product. Express your answer as an equation showing the area of the rectangle as a product and as a sum.

$$(2x + 1)(x - 3)$$

A generic rectangle can be used instead of individual algebra tiles. Draw the generic rectangle and label the sides. Then, determine the areas of the interior rectangles. Write the area as a sum, $-6x + (-3) + 2x^2 + x$. Then, rewrite the expression, $2x^2 + (-5x) - 3$. Write an equation showing that both representations of the area are equal.

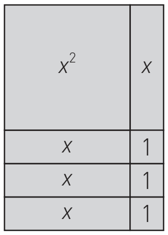


$$\underbrace{(2x + 1)(x - 3)}_{\text{area as a product}} = \underbrace{2x^2 - 5x - 3}_{\text{area as a sum}}$$

Practice

Consider the following sets of algebra tiles and generic rectangles. Write an equation for each showing the area as a product and as a sum.

1.



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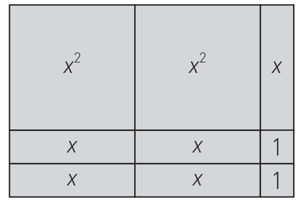
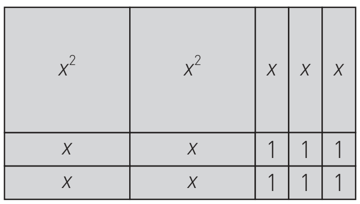
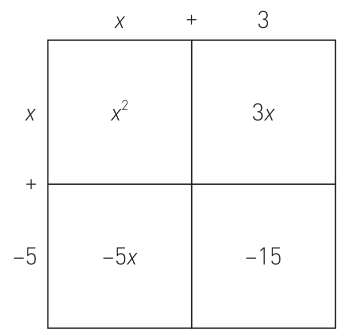


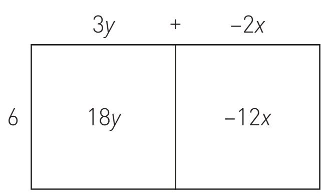
3.



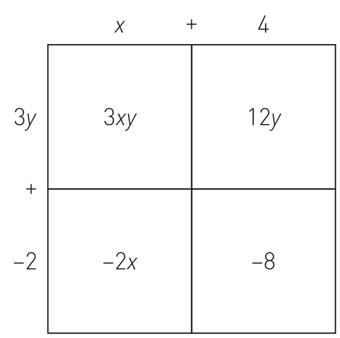
4.



5.



6.



Multiply each expression. Use algebra tiles or a generic rectangle to support your answer.

7. $(3x + 2)(2x + 7)$

8. $(2x - 1)(3x + 1)$

9. $(2x)(x - 1)$

10. $(2y - 1)(4y + 7)$

11. $(y - 4)(y + 4)$

12. $(y)(x - 1)$

13. $(3x - 1)(x + 2)$

14. $(2y - 5)(y + 4)$

15. $(3y)(x - y)$

16. $(3x - 5)(3x + 5)$

17. $(4x + 1)^2$

18. $(x + y)(x + 2)$

19. $(2y - 3)^2$

20. $(x - 1)(x + y + 1)$

21. $(x + 2)(x + y - 2)$

Answers

1. $(x + 3)(x + 1) = x^2 + 4x + 3$
2. $(x + 2)(2x + 1) = 2x^2 + 5x + 2$
3. $(x + 2)(2x + 3) = 2x^2 + 7x + 6$
4. $(x - 5)(x + 3) = x^2 - 2x - 15$
5. $6(3y - 2x) = 18y - 12x$
6. $(3y - 2)(x + 4) = 3xy - 2x + 12y - 8$
7. $6x^2 + 25x + 14$
8. $6x^2 - x - 1$
9. $2x^2 - 2x$
10. $8y^2 + 10y - 7$
11. $y^2 - 16$
12. $xy - y$
13. $3x^2 + 5x - 2$
14. $2y^2 + 3y - 20$
15. $3xy - 3y^2$
16. $9x^2 - 25$
17. $16x^2 + 8x + 1$
18. $x^2 + 2x + xy + 2y$
19. $4y^2 - 12y + 9$
20. $x^2 + xy - y - 1$
21. $x^2 + xy + 2y - 4$

Multiple Methods for Solving Equations

GUIDED SKILL SUPPORT

Overview

Equations may be solved in a variety of ways, including **Rewriting**, **Looking Inside**, and **Undoing**. Rewriting is used to analyze an equation from another perspective. Looking Inside involves focusing on part of the equation to make more sense of the whole equation. Undoing uses inverse operations to isolate the variable.

For additional information about solving equations, see the Methods & Meanings box in Lesson 3.3.3.

Examples

For problems 1 through 5, solve each of the equations for *x*.

1. Solve: $\frac{x}{2} + \frac{x}{5} = 3$

Use **Rewriting** because the equation contains more than one fraction.

$\frac{x}{2} + \frac{x}{5} = 3$	
$10\left(\frac{x}{2} + \frac{x}{5}\right) = 3 \cdot 10$	Multiply by a common denominator.
$5x + 2x = 30$	Rewrite using Fraction Busters.
$7x = 30$	Combine like terms.
$x = \frac{30}{7}$ or ≈ 4.29	Divide both sides by 7.

The solution can be checked by substituting the value of *x* into the original equation.

$$x = \boxed{\frac{30}{7}}$$

$$\frac{x}{2} + \frac{x}{5} = 3$$

$$\frac{\frac{30}{7}}{2} + \frac{\frac{30}{7}}{5} = 3$$

$$70\left(\frac{30}{14} + \frac{30}{35}\right) = 3 \cdot 70$$

$$150 + 60 = 210$$

$$210 = 210$$

true

2. Solve: $\sqrt{x} + 1 = 6$

Use **Looking Inside**, because the equation involves taking the square root of a variable.

$$\sqrt{x} + 1 = 6$$

$? + 1 = 6$ Use Looking Inside to make sense of the radical.

$\sqrt{x} = 5$ Think, "What plus 1 equals 6?"

$x = 25$ Think, "The square root of what number is 5?"

Check that the solution makes the original equation true.

$$x = 25$$

$$\sqrt{x} + 1 = 6$$
$$\sqrt{25} + 1 = 6$$
$$5 + 1 = 6$$
$$6 = 6$$

true

3. Solve: $\frac{2}{3}x + 1 = 7$

Use **Undoing** for equations that can be solved in one or two steps and when another strategy might not be helpful.

$$\frac{2}{3}x + 1 = 7$$

$\frac{2}{3}x = 6$ Subtract 1 from both sides of the equation.

$\left(\frac{3}{2}\right)\frac{2}{3}x = 6\left(\frac{3}{2}\right)$ Multiply both sides of the equation by $\frac{3}{2}$.

$$x = 9$$

Check that the solution makes the original equation true.

$$x = 9$$

$$\frac{2}{3}x + 1 = 7$$
$$\frac{2}{3}(9) + 1 = 7$$
$$6 + 1 = 7$$
$$7 = 7$$

true

4. Solve: $(x - 3)(x + 6) = x^2 - 12x - 6$

Use **Rewriting** when terms can be multiplied.

Rewrite the left side of the equation by multiplying the binomials using a generic rectangle. Then, use algebra to solve the equation for x.

	x	+ 6
x	x^2	$6x$
+ -3	$-3x$	-18

$(x - 3)(x + 6) = x^2 - 12x - 6$	
$x^2 - 3x + 6x - 18 = x^2 - 12x - 6$	Rewrite the equation.
$3x - 18 = -12x - 6$	Subtract x^2 from both sides of the equation.
$15x - 18 = -6$	Add $12x$ to both sides of the equation.
$15x = 12$	Add 18 to both sides of the equation.
$x = \frac{4}{5}$	Divide both sides of the equation by 15.

Check that the solution makes the original equation true.

$$x = \frac{4}{5}$$

$$(x - 3)(x + 6) = x^2 - 12x - 6$$

$$\left(\frac{4}{5} - 3\right)\left(\frac{4}{5} + 6\right) = \left(\frac{4}{5}\right)^2 - 12\left(\frac{4}{5}\right) - 6$$

$$\left(-2\frac{1}{5}\right)\left(6\frac{4}{5}\right) = \frac{16}{25} - \frac{48}{5} - 6$$

$$-14\frac{24}{25} = -14\frac{24}{25}$$

true

5. Solve: $10^{-5x+7} = 100^x$

Use **Rewriting** to make the bases the same. When the bases are the same, the exponents must be equal to make the equation true. Use **Looking Inside** the exponents to solve the equation.

$10^{-5x+7} = 100^x$	
$10^{-5x+7} = (10^2)^x$	Rewrite the right side as base 10.
$-5x + 7 = 2x$	Look Inside to focus on the exponents.
$7 = 7x$	Add $5x$ to both sides of the equation.
$1 = x$	Divide both sides of the equation by 7.

Check the solution.

$$x = 1$$

$$10^{-5x+7} = 100^x$$
$$10^{-5(1)+7} = 100^{(1)}$$
$$10^2 = 100$$
$$100 = 100$$

true

Practice

Solve each equation. Identify all possible solutions for each equation.

1. $\frac{3}{2}x + \frac{1}{2} = 5$

2. $(x - 1)(x + 3) = x^2 - 2x + 13$

3. $\sqrt{2x+5} = 10$

4. $625 = 5^{-6x+5}$

5. $3(2x - 7) = -21$

6. $2^{4x-1} = 32$

7. $\sqrt{x} - 3 = 7$

8. $(x + 1)^2 = 81$

9. $\frac{x}{2} - \frac{x}{5} = 3$

10. $3^{x+3} = 27$

11. $2\sqrt{x-3} = 8$

12. $0.04(x - 1) = 0.16$

13. $400 + 300x = 1{,}300$

14. $10 - (x + 7) = 5$

15. $\frac{m}{3} - \frac{2m}{5} = \frac{1}{5}$

16. $x^2 + 5 = 4$

17. $x(3x - 1) + 8 = (3x + 1)(x - 5)$

18. $0.2x - 0.4 = 1.2$

19. $(y - 1)^2 = 9$

20. $20{,}000 - (3{,}000x) = 10{,}000$

Answers

1.	$x = 3$	**2.**	$x = 4$	**3.**	$x = \frac{95}{2} = 47.5$
4.	$x = \frac{1}{6}$	**5.**	$x = 0$	**6.**	$x = 1.5$
7.	$x = 100$	**8.**	$x = 8$ or -10	**9.**	$x = 10$
10.	$x = 0$	**11.**	$x = 19$	**12.**	$x = 5$
13.	$x = 3$	**14.**	$y = -2$	**15.**	$m = -3$
16.	no solution	**17.**	$x = -1$	**18.**	$x = 8$
19.	$y = 4$ or -2	**20.**	$x = \frac{10}{3}$		